SOVIET PHYSICS

Acoustics

Volume 2 Number [illegible]

A Translation
of the
Journal of Acoustics
of the
Academy of Sciences of the USSR

Published by the
AMERICAN INSTITUTE OF PHYSICS
INCORPORATED

[illegible] of the USSR

[illegible] of Physics Advisory Board on Russian Translations

[illegible]

This volume is published quarterly. Volume 1 will contain all of the scientific material published in the original Russian journal during the calendar year 1955, and Volume 2 will contain all of the scientific material published in the original Russian journal during the calendar year 1956.

Subscription Prices:

Per year, beginning with Volume 2 (4 issues)

United States and Canada	$20.00
Elsewhere	22.00

*Back Numbers (beginning with Volume 2)

Single copies	$ 6.00

*Volume 1, consisting of two issues, is available only as a set at the yearly subscription price.

Subscriptions should be addressed to the American Institute of Physics, 57 East 55th Street, New York 22, New York.

Copyright by American Institute of Physics [illegible]

SOVIET PHYSICS

Acoustics

SOVIET PHYSICS – ACOUSTICS VOLUME 2, NUMBER 3

PROPAGATION OF SOUND IN INHOMOGENEOUS MEDIA*

(SURVEY)

L. M. Brekhovskikh

The indicated topic of this paper is a very broad one. It includes the propagation of sound in solid bodies, liquids and gases. The inhomogeneities may be caused by foreign inclusions, by temperature, velocity and concentration gradients, and by the granular structure of the substance. Such different phenomena as "super-long-range" propagation of sound in the atmosphere and in the ocean over distances of hundreds and thousands of kilometers, the scattering of sound by inhomogeneities of a turbulent nature in the atmosphere, the absorption of sound waves in blankets which are placed upon the walls of special sound-absorbent chambers that are used for acoustical measurements, the propagation of ultrasonic waves in polycrystals and in emulsions, etc. – all of this is included in the propagation of sound in inhomogeneous media. Also included in this subject is the scattering of sound waves at uneven surfaces and at surfaces whose impedance varies from point to point.

However, it is impossible to encompass such an extensive field in a brief paper. Included in this paper are a very great number of individual physical phenomena as well as the theoretical concepts which must be used to describe these phenomena. Therefore we shall limit ourselves to examining two aspects of this subject: the propagation of sound waves in so-called laminarly inhomogeneous media, and the propagation of sound waves in media where the inhomogeneities are of a statistical character.

A laminarly inhomogeneous medium is characterized by the fact that its properties (for example, density and sound velocity) depend upon some one of the coordinates and remain constant on surfaces which are orthogonal to this coordinate. From here on we shall, for the sake of simplicity, examine the case of a plane laminar medium, where it is convenient to choose a rectangular system of coordinates.

During the past ten years a great number of papers have been devoted to the problem of wave propagation in laminar media. Notwithstanding the extremely great mathematical difficulties, the general picture of this phenomenon has at the present time been basically clarified.

The atmosphere and the ocean (when they are somewhat idealized) are typical laminarly inhomogeneous media. The problem which we pose here also has considerable significance with respect to the design of sound-absorbent blankets for special "dead" acoustical measuring chambers. In investigating the propagation of elastic

---

*** A survey read at the Second International Acoustics Congress, June 21, 1956 in Cambridge, Massachusetts, U.S.A.**

waves in the solid shell of the earth, we necessarily encounter both continuous and sharply discontinuous variation of the parameters of the medium with a variation in depth.

The accurate solution of the problem of wave propagation in a medium with continuously varying parameters has been obtained only for certain types of variation functions. The great majority of papers published during the last few years have been devoted to approximate or empirical methods of analyzing this problem for any parameter variation functions. Substantial success has been achieved in this direction. Instead of using a wave equation for the potential of the sound field it usually proves to be rational to make immediate use of the equation for the reflection coefficient; in the final analysis we are exclusively interested in this latter equation.

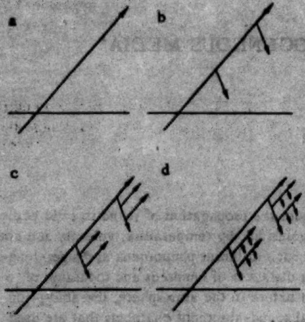


Fig. 1

The equation for the reflection coefficient proves to be an equation of the Ricatti type. One of the most practical approximate methods for solving this type of equation can well be applied to the case which arises when the properties of the medium vary comparatively slowly in space. Then we take as a zero approximation the solution which is in the form of a wave propagated in the medium without reflection (Fig. 1, a). In the first approximation we take into account that this wave initiates elementary reflected waves at every point (Fig. 1, b); the reflection coefficient is equal to zero under these conditions. In the second approximation we take into account that these reflected waves in turn generate waves which are propagated in the same direction as the primary wave (Fig. 1, c); this is caused by the inhomogeneity of the medium. In the third approximation these latter waves generate new waves (Fig. 1, d) which add to the reflected wave, etc.

There is still another approximate method which can well be applied to the case which arises when the parameters of the medium vary at any arbitrary rate, but where the layer from which the investigated reflection takes place is sufficiently thin. This method is used especially often in those cases where it is necessary to determine to what extent the reflection coefficient varies, or when we have a somewhat blurred boundary separating two media instead of a sharply defined boundary (i.e., when there is a transition layer).

The theory of sound propagation shows that in an inhomogeneous medium it is impossible, generally speaking, to separate the total sound field into incident and reflected waves. This can easily be understood from Fig. 1. Such a separation is not single-valued. Gans [1], who was one of the first to analyze the laws governing the complete reflection of waves in an inhomogeneous medium (1915), introduced a basic error in this regard. In particular, he posed the problem of finding the direction of the rays at the recursion point of the wave. Usually a wave which has been refracted and completely reflected in an inhomogeneous medium is represented by a ray in the manner depicted in Fig. 2 by the solid line. Gans, having found a completely correct expression for the total field in the region where the wave performs its recursion, arbitrarily separated this field into incident and reflected waves, represented each of them by a ray and found that these rays formed an angle at the point of complete reflection of the wave (the dotted line in Fig. 2). In actual fact, however, this assertion

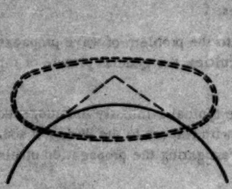


Fig. 2

has no meaning; in general (as we can show) it is true that a ray cannot be constructed in the region enclosed by the double dotted line in Fig. 2, since in this region the ray theory is not applicable.

Speaking of reflection from inhomogeneous media, we should dwell upon an interesting remark made by Kofink [2]. He pointed out that for a wave with a specified frequency it is always possible to find any number of arbitrary inhomogeneous layers in which the parameters vary in such a manner with respect to the thickness of the layer that the wave will completely pass through these layers without reflection. He also provided an extremely simple method of selecting such layers.

Up to now we have examined media whose parameters vary continuously. However, media in which the parameters can vary only discontinuously also belong to the class of inhomogeneous media. The practical importance of the problem concerning wave propagation in media of this type is extremely great. As was shown in the paper by Beranek and his coworkers [3], it is possible to obtain effective sound insulation over a wide band of frequencies while utilizing structures which are composed of combinations of layers. The degree of sound insulation is determined by the reflection coefficients and by the transparence coefficients.

For the purpose of making a rational selection of "translucent" layers for acoustic lenses it is also necessary to be able to calculate the reflection coefficient of the system of layers. A corresponding theory has been developed, in particular by B. D. Tartakovskii [4-5], and although the computing formulas are complex it is not difficult to design a system of "translucent" layers which would assure, for example, the complete transparency of the boundary between aluminum and air over a rather wide band of frequencies. S. M. Rytov [6] examined the extreme case when the thickness of the individual layers is extremely small in comparison to a wavelength. It proved to be true that a medium consisting of a large aggregate of such layers can be examined as a medium which is homogeneous but is also anisotropic. It was shown that such a medium is equivalent to a crystal with hexagonal symmetry and that it is characterized by five elasticity constants.

Let us turn to the case of wave propagation where the waves are radiated by a nondirectional point source. The great mathematical difficulties of this problem arise because of the difference in the symmetry of the medium and the symmetry of the radiated wave. The medium is plane laminar and the source radiates a spherical wave. However this very fact leads us to the most rational method for examining the problem. It is necessary to decompose the spherical wave into plane waves, the more so because the behavior of plane waves has already been examined above. The method proves to be sufficiently fruitful both in the case of media with smoothly varying properties and in cases where separation boundaries are present. We shall dwell on the basic results which are obtained when it is assumed that the inhomogeneities of the medium are concentrated in a certain finite region in the direction of the "vertical" coordinate $\underline{z}$, and that we have homogeneous media when $z \to \pm \infty$.

The sound field which is propagated in the laminar medium in this case can be subdivided into three parts.

The first and evidently the most important part is the aggregate of a finite number of so-called normal waves. Each normal wave has its own phase velocity of propagation along the layers and its own characteristic dependence of amplitude upon the coordinate $\underline{z}$. The dependence of the amplitude upon the horizontal distance $\underline{r}$ is given by the law $\sim \frac{1}{\sqrt{r}} e^{-\beta_n r}$. Here the factor $1/\sqrt{r}$ corresponds to a cylindrical spreading of the wave, and the factor $e^{-\beta_n r}$ (which depends upon the wave number) describes the attenuation.

The second part can be called the "remainder normal waves." Although their number is infinitely large, these waves diminish rapidly in amplitude as they are propagated away from the source. They are important only in the immediate vicinity of the radiator.

The third part consists of two waves which are called lateral waves. When there is no absorption of sound energy in the medium, the amplitude of the lateral waves diminishes according to the law $1/r^2$ when they are sufficiently far away from the source.

The propagation velocity and attenuation of each normal wave can be easily found in principle, but great difficulties may be encountered in performing practical calculations.

Let us assume that there is no shear resistance in the medium. Then the characteristics of the entire system of normal waves (including the "remainder" waves) can be found from the equation

$$V_1 V_2 = 1. \tag{1}$$

Here V_1 and V_2 are the reflection coefficients for the reflection of a plane wave from the inhomogeneous half-spaces $z < z_1$ and $z > z_1$ respectively, where $z = z_1$ is some arbitrarily selected level. The quantities V_1 and V_2 are functions of the angle of incidence θ of the wave. Solving Equation (1), we find an infinite number of solutions $\theta_1, \theta_2, \ldots$, each of which will be associated with an individual normal wave. In the U.S.A. Ivan Tolstoy [7] made very skilled use of Equation (1).

It is essential to note that any value of z_1 may be selected in order to determine the characteristics of a system of normal waves. An exception is the case when individual portions of the medium are separated from one another by absolutely nontransparent boundaries. In that case we shall have two or more systems of normal waves which are independent of one another. The characteristics of each of these systems must then be found independently.

The generalization of Equation (1) to include the case of solid layers does not present any complications. In that case the coefficients of converting one set of waves into the other sets will be included in the expression, in addition to the reflection coefficients of longitudinal and shear plane waves.

For a given medium, the number of normal waves which are associated with the first portion of the field (discussed above) is the greater, the higher the frequency. When the frequency is increased we encounter an extremely interesting phenomenon – new normal waves are "generated" at certain definite frequencies. The behavior of the lateral waves at these frequencies also has interesting features.

Practically speaking, all cases of wave propagation in laminarly inhomogeneous media are rather sharply separated into two classes which I would call "wave-duct" and "anti-wave-duct" propagation. In wave-duct propagation the normal waves are attenuated very slowly with distance from the source. In these cases the sound can be propagated over very large distances. It is precisely this fact which explains the possibility of registering the sounds of powerful explosions at distances of up to a thousand and more kilometers.

Still more amazing is the phenomenon of the wave-duct propagation of sound in the ocean. The sound of an explosion of a TNT charge weighing several kilograms can be fixed at a distance of more than 5,000 kilometers. It requires more than an hour for the sound pulse to traverse this distance. This phenomenon is as amazing, says the author of one paper, as if sounds produced by an orchestra at the beginning of a concert were retained in the hall until the end of the concert.

Although the phenomenon of wave-duct propagation can rarely be described accurately by means of simple ray methods, it is nevertheless true that such representation is sufficiently adequate for the purpose of a qualitative explanation of the phenomenon.

Let the dependence of the sound velocity c upon the vertical coordinate z be of the form depicted in Fig. 3, a. When $z = z_0$, we have a minimum for the sound velocity. If a sound emitter O is located at this level, then a definite class of rays which are emitted from the source at slip angles which are not too great returns to this level again and again during further propagation. The sound energy at large distances from the source will be basically concentrated in some layer which is close to the level at which the sound velocity is minimum. This layer is called an underwater sound channel in the case where the sound is propagated in the ocean. Such a channel arises due to the combined effect of the temperature gradient and the hydrostatic pressure gradient. In the atmosphere an analogous wave-duct is formed due to the temperature inversion at altitudes of 30-50 kilometers.

Wave-duct propagation takes place not only when the radiator is located at the level where the sound velocity is minimum (or close to it), but also in the case when the radiator is located completely outside of the wave-duct, for example, when $z = z'$ (Fig. 3, a). In that case the sound energy is caught up by the wave-duct by means of diffraction and is propagated along the wave-duct in the normal manner from then on. Ray representation is clearly no longer applicable for the purposes of explaining this phenomenon.

When sound waves are propagated in wave-ducts a very interesting phenomenon may occur — the focusing of the sound waves. Usually the phenomenon of wave focusing is associated with the effects produced by special lens and mirror systems; however when waves are propagated in inhomogeneous media this phenomenon is evidenced quite often.

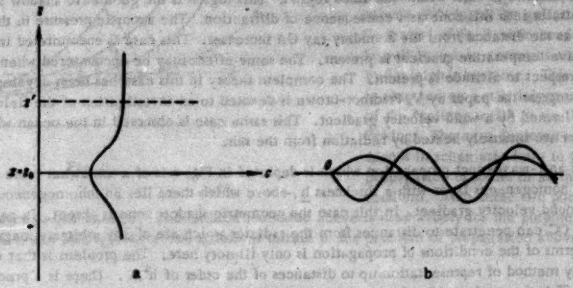


Fig. 3

A wave-duct, as a rule, has a certain critical frequency. At frequencies below the critical frequency waves cannot be retained inside the wave-duct since the duct becomes too narrow for them. Ray methods, naturally, prove to be poorly suited for determining the critical frequency.

The complete wave theory of wave-duct propagation is quite complex. However, it should be noted that Iu. L. Gazarian [8] has recently examined one case of wave-duct propagation that evidences extremely characteristic features and can be quite easily solved. "The shape of the wave-duct" (i.e., the dependence of the sound velocity upon the coordinate z) was selected in such a way that it corresponded to the so-called "Epstein Layer". We should also mention that V. A. Fok [9] pointed out the very close analogy between wave-duct propagation of waves in an inhomogeneous medium and the propagation of a wave packet in quantum mechanics. He showed the complete correspondence between the mathematical formulations of these two problems.

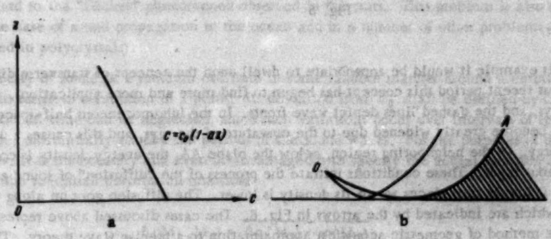


Fig. 4

In the case of anti-wave-duct propagation the normal waves prove to be strongly attenuated, and the sound intensity may be extremely small at a sufficiently large distance from the source. A typical example of anti-wave-duct propagation is the case depicted in Fig. 4. An absolutely reflecting boundary is located in the plane $z = 0$. In the half-space $z > 0$, the velocity of sound decreases according to a linear relationship as the distance away from the boundary increases. The path traversed by the rays in this case is depicted in Fig. 4, b. Not one ray emanating from the radiator O falls within the lined region. This region is the geometric shadow zone. The sound energy only penetrates into this zone as a consequence of diffraction. The acoustic pressure in the shadow zone decreases rapidly as the distance from the boundary ray OA increases. This case is encountered in the atmosphere when a negative temperature gradient is present. The same effect may be encountered when a wind velocity gradient with respect to altitude is present. The complete theory in this case has been developed by Pekeris [10]. At this Congress the paper by V. Pridmor-Brown is devoted to an investigation of the field in the shadow region which is formed by a wind velocity gradient. This same case is observed in the ocean when the upper layers of the water are intensely heated by radiation from the sun.

The other case of anti-wave-duct propagation which is depicted in Fig. 5 is of a somewhat more complex nature. Here there is a homogeneous layer with a thickness h, above which there lies an inhomogeneous half-space with a negative sound velocity gradient. In this case the geometric shadow zone is absent. In particular, rays of the type OB and OC can penetrate to distances from the radiator which are of any arbitrary magnitude. However "success" in terms of the conditions of propagation is only illusory here. The problem is that we can only make use of the ray method of representation up to distances of the order of h^2/λ. There is a precise wave theory for such a case. This theory shows that for propagation in a homogeneous layer (for example, along the ray OB) a specific attenuation of the wave occurs, which is associated with the withdrawal of sound energy into the inhomogeneous half-space.

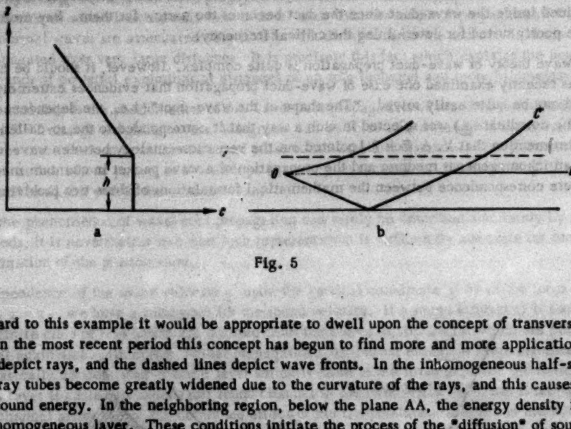


Fig. 5

With regard to this example it would be appropriate to dwell upon the concept of transverse diffusion of sound energy (in the most recent period this concept has begun to find more and more application). The solid lines in Fig. 6 depict rays, and the dashed lines depict wave fronts. In the inhomogeneous half-space (above the plane AA) the ray tubes become greatly widened due to the curvature of the rays, and this causes a drop in the density of the sound energy. In the neighboring region, below the plane AA, the energy density is considerably greater in the homogeneous layer. These conditions initiate the process of the "diffusion" of sound energy from places where its density is higher into places where its density is lower. The diffusion goes on along the wave fronts in the directions which are indicated by the arrows in Fig. 6. The cases discussed above represent a successive (following the method of geometric acoustics) approximation to a precise wave theory. They can be developed into a rigorous mathematical form. G. D. Maliuzhinets is developing these ideas in detail.

Speaking of the propagation of sound waves in laminarly inhomogeneous media, it is useful to point out the fact that theoreticians in a number of cases find it convenient to use the concept of "artificial" inhomogeneous media. Thus, in particular, the investigation of wave propagation in the vicinity of a curved wall (Fig. 7, a) reduces to propagation in a half-space with a plane boundary but with a sound velocity which increases according to a linear relationship with distance from the boundary (Fig. 7, b). The ray picture for this case is depicted in Fig. 7, c. The case depicted in Fig. 7, a corresponds to the phenomenon of "whispering galleries" in the Cathedral of St. Paul in London; this phenomenon is well known in acoustics and was explained by Rayleigh in his time. This same phenomenon is observed on a still larger scale in the vicinity of the "talking wall" in the Tomb of Heaven in Peking. The case depicted in Fig. 7, c corresponds in schematic form to the wave-duct propagation of sound in an inhomogeneous medium. Thus these two phenomena prove to be closely related to one another. The method indicated above is used widely for the purpose of reducing the problem of radiowave propagation above spherical terrain to the problem of propagation above flat terrain.

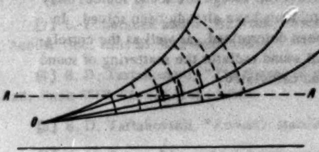


Fig. 6

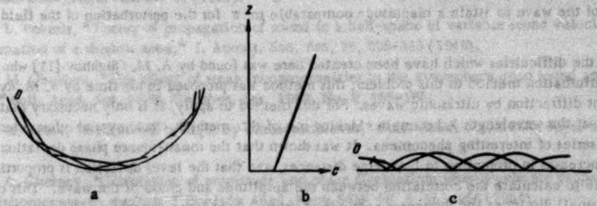


Fig. 7

Let us now examine the propagation of sound in a medium with inhomogeneities which are of a statistical character. In the atmosphere such inhomogeneities arise as a consequence of turbulence. They were discovered long ago with regard to the "flicker" phenomenon observed in the stars. This problem is also of substantial significance in the case of sound propagation in the ocean and in a number of other problems (for example, when sound is propagated in polycrystals).

Let the velocity of sound in a medium oscillate in some random manner around a certain average value c_0 when viewed in terms of a function of a point. Its deviation from c_0 shall be defined by $\Delta c = \Delta c\,(x, y, z)$. The variation of Δc in time can be neglected, since such variations occur over intervals of time which, in their order of magnitude, substantially exceed the periods of the sound waves. For the purposes of the theory which we shall discuss below, it is unnecessary and even disadvantageous to specify the actual function Δc. Rather it is essential to know the so-called correlation function:

$$B(M_1, M_2) = \overline{\Delta c\,(M_1) \cdot \Delta c(M_2)},$$

where M_1 and M_2 are two arbitrary points in the medium. Usually the analysis is limited to the case where the correlation function depends solely on the distance r between the points in the medium. Assuming r = 0, we also obtain the important quantity $\overline{(\Delta c)^2}$ – the mean-square value of the sound velocity fluctuations. Usually the analysis is limited to the case where the fluctuations are relatively small (i.e., when $\left|\frac{\Delta c}{c_0}\right| << 1$.

After the properties of the medium have thus been specified, the theory is faced with the problem of expressing the characteristics of the sound field in terms of these properties (even though the sound sources may be of the simplest types). At the present time a number of essential problems have already been solved. In particular, the fluctuations in acoustic pressure level and phase have been determined, as well as the correlations of these fluctuations at various points. In the case of a directional sound radiator the scattering of sound waves from the "illuminated" region into the remaining space has been calculated.

The so-called "wave parameter"

$$D = \frac{\lambda L}{2l^2},$$

plays an essential role in the theory, where λ is the acoustic wavelength, L is the distance traversed by the wave in the inhomogeneous medium, ℓ is the average dimension of the inhomogeneities of the medium ("the correlation radius" for the quantity Δc). When D << 1 we may apply ray acoustics. At large distances it is necessary to take diffraction at the inhomogeneities of the medium into account. The problem is also made more complex because the usual perturbation method (which assumes that $\varphi = \varphi_0 + \varphi_1$ for the sound potential and that $\varphi_1 << \varphi_0$) has a very narrow applicability in the case being examined. In fact, it is only necessary for the phase perturbation of the wave to attain a magnitude comparable to π for the perturbation of the field to cease being small.

A way out of the difficulties which have been created here was found by A. M. Obukhov [11] who successfully applied the perturbation method to this problem; this method was proposed in his time by S. M. Rytov [12] in the theory of light diffraction by ultrasonic waves. For this method to apply, it is only necessary that the phase perturbations at the wavelength λ be small. Making use of this method, Obukhov and other scientists calculated a whole series of interesting phenomena. It was shown that the mean-square phase deviation from the unperturbed value increases in proportion to L at large distances, and that the level deviation is proportional to L^3. It is also possible to calculate the correlation between the amplitude and phase of the wave. This correlation is constant at small distances and decreases approximately as 1/L at large distances. The correlation of the sound-field fluctuations at two points lying on a straight line perpendicular to the direction of wave propagation was also calculated. The fact that this correlation extends approximately to the same distance as does correlation of the quantity Δc proved to be extremely nontrivial.

A proportion of the indicated results is due to L. A. Chernov [13]. It is he who has recently performed interesting calculations of the process of wave focusing for the case when waves are propagated in media with statistical inhomogeneities; he has substantially refined the results available earlier. At one of the sessions a paper on the propagation of sound waves in media with inhomogeneities of a random character will be made by V. A. Krasil'nikov who also has obtained many interesting results in this field.

Thus we have examined two of the most important cases of sound propagation in an inhomogeneous medium – propagation in a medium with regular inhomogeneities of a laminar character and propagation in a medium with inhomogeneities of a random character. The theory of these two cases is far from being fully developed. Acoustical scientists in all countries are extremely interested in these problems, and numerous papers which appear in various journals bear witness to this. However another, more difficult problem of sound propagation is on the order of the day; this problem involves sound propagation in a medium where inhomogeneities of both indicated types are present simultaneously. It is precisely such a medium which is most representative of actual media, and, in particular, of the atmosphere. The simple and naive concept which assumes that the effects of these inhomogeneities upon sound propagation are additive and independent has now ceased to satisfy many scientists.

LITERATURE CITED

[1] R. Gans, "Fortpflanzung des Lichts durch ein inhomogenes Medium," Ann. Physik 74, 709 (1915).

[2] W. Kofink, "Reflexion elektromagnetischer Wellen an einer inhomogenen Schicht," Ann. Physik 1, 119-124 (1947).

[3] L. L. Beranek and G. A. Work, "Sound transmission multiple structures containing flexible blankets," J. Acoust. Soc. Am. 21, 419-428 (1949).

[4] B. D. Tartakovskii, "The theory of propagation of plane waves through homogeneous layers," Doklady Akad. Nauk SSSR 71, 465-468 (1950).

[5] B. D. Tartakovskii, "Acoustic transition layers," Doklady Akad. Nauk SSSR 75, 29-32 (1950).

[6] S. M. Rytov, "Acoustical properties of a thinly laminated medium," Acoust. J. 2, 1, 71-83 (1956) [Soviet Physics — Acoustics, p. 67].

[7] I. Tolstoy, "Dispersion and simple harmonic point sources in wave ducts," J. Acoust. Soc. Am. 27, 897-907 (1955).

[8] Iu.L.Gazarian, "The problem of wave duct propagation in inhomogeneous media," Acoust. J. 2, 2, 133-136 (1956) [Soviet Physics — Acoustics, p. 134].

[9] V. A. Fok, "Theory of radiowave propagation from a raised dipole in inhomogeneous atmosphere," Bull. Acad. Sci. USSR, Phys. Ser. 14, 70 (1950).

[10] C. L. Pekeris, "Theory of propagation of sound in a half-space of variable sound velocity under conditions of formation of a shadow zone," J. Acoust. Soc. Am. 18, 295-315 (1946).

[11] A. M. Obukhov, "The effect of weak inhomogeneities in the atmosphere upon sound and light propagation," Bull. Acad. Sci. USSR, Geophys. Ser. 1953, No. 2, 155-165.

[12] S. M. Rytov, "Diffraction of light by ultrasonic waves," Bull. Acad. Sci. USSR, Phys. Ser. 1937, No. 2, 223.

[13] L. A. Chernov, "Correlation of amplitude and phase fluctuations when waves are propagated in a statistically inhomogeneous medium," Doklady Akad. Nauk SSSR 98, 6, 953-956 (1954).

Acoustics Institute of the Acad. Sci. USSR
Moscow

Received June 6, 1956.

INSTRUMENT FOR MEASURING THE ELASTIC MODULI OF CRYSTALS

K. S. Aleksandrov and O. V. Nosikov

The paper describes an ultrasonic instrument for measuring the elastic moduli of crystals; the instrument is based upon measuring the propagation velocity of brief pulses of longitudinal and transverse elastic vibrations in specimens with small dimensions. The unknown velocity is determined by the method of superimposing two pulses upon the screen of a cathode ray tube – the two pulses are the pulse which has passed through the specimen and the pulse which has passed through a liquid in which the velocity of sound is known with sufficient accuracy. The liquid used for this measurement is an aqueous solution of ethyl alcohol which has a small temperature coefficient of velocity.

Data is given on the measurement of the elastic moduli of a monocrystal of dihydrophosphate of ammonia; the data for these moduli agrees with the data obtained experimentally by other authors.

The resonance, optical and pulse methods are at the present time the most common dynamic methods for measuring the elasticity constants of crystals. The resonance method is the most accurate and at the same time is the most cumbersome method, since it requires thorough processing of a large number of specimens. The optical method is so far mainly applied to transparent crystals.

The pulse method of measuring the elastic moduli of monocrystals, although it has its shortcomings (the first of these which should be mentioned is the lower accuracy of the measurements and the relatively large dimensions of the specimens), offers a number of advantages in comparison with other methods. The most substantial advantage of the pulse method is the simplicity of the measurements and the lesser need for a large number of specially oriented specimens.

For many classes of crystals it is possible to find directions in which purely longitudinal or purely transverse waves are propagated. The propagation velocities of these waves, when they are measured by the pulse method, are associated with the elastic moduli of the crystal by means of a simple relationship [1, 2]. However, only when the crystals belong to the cubic system are the number of such "special" directions sufficient for determining all of the three independent elastic moduli which can be measured on one specimen. For crystals with a lower degree of symmetry the moduli c_{ik} $(i = k)$ can be determined from measuring the propagation velocities in the "special" directions; however the moduli c_{ik} $(i \neq k)$ are found from measurements which are taken in arbitrary crystallographic directions and are thus less accurate. Judging by the data available in the literature, the possibilities of the pulse method are at the present time limited to nineteen crystallographic classes [3].

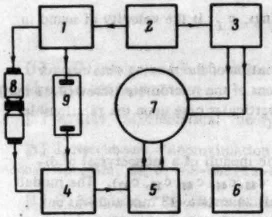
In this paper we describe the principle of operation and the possibilities of application of an ultrasonic pulse velocity-meter which was designed and manufactured in the laboratory of ultrasonic defectoscopy of the V. I. Ul'ianov (Lenin) Leningrad Electrical Engineering Institute under the direction of Corresponding Member of the Academy of Sciences, USSR, S. Ia. Sokolov.*

---

* In addition to the authors, Engineers V. A. Shchukin, E. V. Pigulevskii and A. S. Golubev took part in the design and manufacture of the instrument.

The principle of operation of this instrument consists of measuring the time which it takes for a brief ultrasonic pulse to pass through a crystalline specimen of specified length. Similar instruments which are referred to in the literature [4] usually use standard radar station units and electronic circuits which measure time intervals. Our problem was the design of a compact instrument for measuring propagation velocities in small specimens. Therefore we selected a circuit which permitted us to compare the time which it took for the pulses to pass through the investigated specimen and through a liquid in which the velocity of sound is known with sufficient accuracy. The comparative method of measurement permitted us to lower the stability requirements of the electronic circuit considerably, and thus it permitted us to simplify the latter circuit.

It is possible to utilize distilled water or an aqueous solution of ethyl alcohol in the capacity of the standard liquid. The temperature dependency of the velocity of sound in water has been repeatedly measured and is well known [5]. The velocity of sound in an aqueous solution of ethyl alcohol which has a specified concentration (for example, 16 cc of alcohol per 100 cc of water) is practically independent of temperature and is also well known [6]. We utilized this liquid in our measurements.



Block diagram of the instrument.

1) high-frequency pulse generator, 2) master generator, 3) sweep oscillator and amplifier, 4) amplifier, 5) paraphase amplifier, 6) supply section, 7) oscillograph section, 8) measuring device, 9) vessel containing the liquid.

The block diagram of the instrument is shown in the figure. The master generator 2 produces rectangular pulses with a repetition frequency of approximately 1,000 cps. The pulses from the master generator trigger a high-frequency pulse generator 1 (consisting of a shock-excited circuit with capacitive storage) and a driven sweep oscillator 3 with a linearizing circuit and a phase-inversion stage. The high-frequency pulse generator (1.67 and 5.0 megacycles) simultaneously send ultrasonic pulses with a duration of 2-10 μ sec to the metering device 8 and the vessel with the liquid 9 by means of quartz piezotransducers. The pulses which have passed through the metering device and through the liquid are received by quartz receivers, are amplified by the wide-band amplifier 4 and are fed to the vertical plates of the cathode ray tube 7 by means of a paraphase amplifier 5. The metering device 8 which is utilized in the instrument has been described in [4]; it is utilized for the measurement of the velocity of compression waves in small specimens. The device consists of a pair of cylindrical rods made out of small-grain steel, which are shifted by means of parallel guides. The rods have plane-parallel end-faces; a quartz plate is glued onto one of the faces and the test specimen is compressed between the rods by means of a special screw. The device has two pairs of rods of different lengths for the purposes of measuring the velocities of longitudinal and transverse elastic waves. X-cut and Y-cut quartz plates are used in the capacity of the ultrasonic radiators and receivers; these quartz dipoles operate at a fundamental frequency of 1.67 megacycles and at a third harmonic frequency of 5.0 megacycles. There are also two X-cut quartz plates in the vessel containing the liquid 9 (the temperature of this liquid is measured by a thermometer). The initial distance between the plates is selected in such a way that the time that is required for a pulse to traverse this distance is equal to the time for a pulse to traverse the length of the rods. By means of a micrometer screw the distance between the radiator and receiver in the liquid can be increased by another 25 mm. Thus the maximum dimension of the specimens is limited by the time required for an ultrasonic pulse to traverse a 25 mm column of liquid.

Purified transformer oil was used to guarantee acoustical contact between the rod and the specimen for longitudinal waves. For shear waves we hit upon a solution of pine tar in acetone after a number of unsuccessful experiments. After a drop of the solution was deposited upon the surface of the specimen, the acetone rapidly evaporated, and a layer of tar which allowed the shear waves to pass through it remained between the metal and specimen.

The process of measuring the velocity of sound in the specimen, both for longitudinal and for transverse waves, consists of the following procedure. Metallic ultrasonic delay lines (rods) are brought together and two pulses are superimposed on the screen of the instrument by turning the micrometer screw; these two pulses are the pulse which has passed through the liquid (the adjustable pulse) and the pulse which has passed through the measuring device (the stationary pulse). For the purpose of achieving accurate superposition, the instrument utilizes a circuit which permits the small sector of the sweep which encompasses the region where the two pulses are located to be extended across the entire screen. The superposition process is carried out in terms of the maximum amplitude of one of the half-cycles of the high-frequency wave (the pulses are not detected). Then the test specimen is inserted between the rods and the pulses are superimposed on the screen in the same manner at the maximum of the same half-cycle of the high-frequency wave.

The sound velocity in the specimen is determined according to the formula

$$c_{sp} = \frac{l}{n_2 - n_1} \cdot c_{\ell},$$

where ℓ is the length of the specimen, n_1 and n_2 are the micrometer readings, c_{ℓ} is the velocity of sound in the liquid at the specified temperature.

Measurements showed that after a certain amount of practice the deviation of the reading obtained for repeated superpositions at a constant temperature did not exceed 1-2 divisions of the micrometer screw (0.01 mm). The error in determining the values of the elastic moduli depends in each particular case upon the relationship between the given elastic modulus and the measured velocity.

For the purpose of calibrating the instrument we determined the elastic moduli of a monocrystal of dihydrophosphate of ammonia which had six independent elastic moduli (c_{11}, c_{33}, c_{44}, c_{66}, c_{12}, c_{13}). The moduli c_{ik} ($i = k$) were determined on the same specimen which had dimensions of 20 × 18 × 18 mm and was cut perpendicular to the crystallographic axes; these moduli were found according to formulas of the form: $c_{ii} = \rho c_{sp}^2$, where ρ is the density of the material which makes up the specimen. For the purposes of determining the moduli c_{12} and c_{13} two more specimens were prepared, and the velocities of the longitudinal and transverse waves in the directions [110] and [101] respectively were measured. The results of the measurements are shown in the table along with measurements obtained by other authors. From the table it is evident that in absolute value our measurements agree fairly well with the data available in the literature [7, 8, 9 and 10].

TABLE

Elastic Moduli of Dihydrophosphate of Ammonia ($c_{ik} \cdot 10^{-11}$ dyne/cm^2)

Modulus	Experiment	Pulse method [7]	Optical method [8]	Resonance method [9]	Average value [10]
c_{11}	6.72 ± 0.04	6.89 ± 0.02	6.17 ± 0.03	6.77	6.61
c_{33}	3.42 ± 0.07	3.35 ± 0.04	3.28 ± 0.02	3.38	3.34
c_{44}	0.830 ± 0.015	0.856 ± 0.003	0.850 ± 0.025	0.868	0.858
c_{66}	0.604 ± 0.008	0.595 ± 0.004	0.592 ± 0.006	0.608	0.598
c_{12}	0.62 ± 0.03	0.40 ± 0.04	0.72 ± 0.14	0.59	0.57
c_{13}	1.95 ± 0.13	1.89 ± 0.06	1.94 ± 1.0	1.99	1.94

The values of the elastic moduli and the errors in our measurements were determined as average values from the results of several (5-7) repeated measurements in the same crystallographic direction (c_{33}, c_{66}) or in two equivalent directions (c_{11}, c_{44}).

The large errors in determining the elastic moduli c_{12} and c_{13} are explained by the fact that the velocity of sound in the directions [110] and [101] depends simultaneously upon the magnitude of several elastic moduli, and that therefore the errors in measuring the remaining moduli are included in the error of measuring these moduli. In addition, the insufficient accuracy of orienting the specimens and the possible inhomogeneities in the monocrystal increased the error of measuring the moduli c_{ik} $(i = k)$.

As can be seen from the data we have cited, the instrument which we have described can find applications in determining the elastic moduli of crystals. As far as the accuracy of its measurements is concerned, inhomogeneities in the monocrystals (such inhomogeneities inevitably arise in the process of growing them) make the accuracy of the measurements obtained with this instrument fully adequate in many cases.

In addition, when the measuring device is slightly altered it is possible to investigate the effect of pressure, temperature and other factors upon the velocity of sound in specimens with a relative accuracy of measurement which attains 0.1-0.4 % for specimens with dimensions of the order of 20 mm and for velocity values of $(2 \text{ to } 6) \cdot 10^5$ cm/sec. A knowledge of the temperature dependencies of the elastic moduli permits us, for example, to use the zero temperature coefficients of the frequency to calculate the orientation of cuts in piezoengineering.

LITERATURE CITED

[1] L. D. Landau and E. M. Lifshits, Mechanics of Continuous Media, (State Tech. and Theor. Lit. Press, Moscow, 1954), p. 756.

[2] A. Liav, Mathematical Theory of Elasticity, (Moscow-Leningrad, 1935), p. 311.

[3] J. Neighbours, "Approximation method for the determination of elastic constants of single crystals," J. Acoust. Soc. Am. 26, 5, 865-869 (1954).

[4] G. Bradfield, "Precise measurements of the velocity and attenuation using ultrasonic waves," Nuovo Cimento Suppl. 1950, No. 2, 162-181.

[5] G. Willard, "Temperature coefficient of ultrasonic velocity," J. Acoust. Soc. Am. 19, 1, 235-241 (1947).

[6] B. Kudriavtsev, "Application of ultrasonic methods in physico-chemical research (State Tech. and Theor. Lit. Press, Moscow-Leningrad, 1952), p. 211.

[7] W. Price and H. Huntington, "Acoustical properties of anisotropic materials," J. Acoust. Soc. Am. 22, 1, 32-36 (1950).

[8] B. Zwicker, "Elastische Untersuchungen an $NH_4H_2PO_4$ und KH_2PO_4," Helv. Phys. Acta 19, 523 (1946).

[9] R. Beckmann, "Contour modes of square plates excited piezoelectrically and determination of elastic and piezoelectric coefficients," Proc. Phys. Soc. B 64, 323-337 (1951).

[10] R. Hearmon, "The elastic constants of piezoelectric crystals," Brit. J. Appl. Phys. 3, 4, 120-124 (1952).

Crystallography Institute of the Acad. Sci. USSR, Moscow
V. I. Vl'ianov (Lenin) Leningrad Electrical Engineering
Institute

Received August 2, 1955

A MODULATION METHOD FOR MEASURING SMALL VOLTAGES IN THE AUDIO-FREQUENCY RANGE

D. K. Balabukha, L. L. Miasnikov and E. N. Plotnikova

The paper describes the application of the modulation method to the measurements of small voltages in the audio-frequency range. The modulation is accomplished by means of periodically varying a capacitance. The paper indicates an approximate method for calculating the input circuit parameters and the parameters of the modulators; appropriate nomograms are provided. A push-pull circuit is proposed for the two modulators which permits simultaneous achievement of large values of modulation depth and large coefficients of voltage transfer. The method analyzed in this paper permits signals to be measured against the background of the intrinsic noise of the metering portion of the apparatus when this noise considerably exceeds the magnitude of the signal; the sensitivity of the instrument has been raised by a factor of 10 in comparison to the devices ordinarily available at the present time.

One of the most important problems of contemporary acoustics and radiophysics is the problem of measuring weak signals in the sonic and ultrasonic frequency range against the background of an intrinsic amplifier noise which exceeds the level of the signals. As we know, the noise of an amplifying unit cannot be lower than the limit determined by the "shot-noise" effect and the thermal noises of the input resistors. The problem of amplifying very weak signals over a wide band of sonic and ultrasonic frequencies has remained open, since the normal methods of selective reception which are used for segregating sinusoidal signals of definite frequencies prove to be ineffective when working in a wide band of frequencies.

In microwave engineering this problem is solved by using the modulation method; an example of this is the modulation method of measuring the radiowave emission from the sun at microwave frequencies. The application of the modulation method for measuring small voltages in the region of low audio frequencies encounters difficulties associated with the close proximity of the modulation frequency to the investigated frequency band, as well as structural difficulties associated with building the modulators. However, these difficulties were successfully overcome. We have developed an audio-frequency microvoltmeter with capacitance modulation, which permits the reception of signals, whose intensity is 10 times lower than the intensity of the intrinsic noise of the apparatus when the input impedance exceeds 100 megohms, to be achieved over the range of audible frequencies. The modulation of the test signal at the input of the microvoltmeter was carried out sinusoidally at a frequency of 24 cps which was lower than the investigated band of frequencies that extended up to 20 kc.

The block diagram of the audio-frequency microvoltmeter which operated according to the modulation method is shown in Fig. 1. The test signal appears across the input terminals a-b. This signal is modulated by means of a capacitive modulator II. The modulated signal is fed through the coupling circuit III into the preamplifier IV. Since a contact potential difference arises between the electrodes of the capacitive modulator during its operation (this contact potential difference is modulated and appears as the input of the amplifier IV together with the test signal) and distorts the signal, a three-stage RC filter V of the T type (which is tuned to the modulation frequency of 24 cps and thus eliminates the interference produced by that frequency) is connected between the main amplifier VI and the preamplifier IV. After modulation and amplification in the basic amplifier VI which has a band-width of 200-20,000 cps, the signal voltage appears across the detector VII. A filter which is tuned to the envelope frequency is connected at the output of the detector.* The modulation-frequency voltage is separated from the noise spectrum by a variable RC-filter-analyzer VIII which is tuned to the modulation frequency of 24 cps. A thermocouple with a galvanometer is connected after the analyzer. The measurement may be performed by the method of substituting a dc voltage for an ac voltage, with the subsequent measurements being performed with dc current so that an increased accuracy is obtained; for this purpose the dc source IX and the potentiometer X are used. Because of the inevitability of fluctuations in the modulation frequency, the pass band of the analyzer cannot be made very narrow. A further increase in selectivity can be achieved by utilizing synchronous detection. For this purpose a synchronous detector XIV is connected at the output of the analyzer instead of the thermocouple; a "base" voltage from another modulator XI (whose capacitance is varied at the same frequency as the capacitance of the basic modulator) is simultaneously fed to the synchronous detector XIV through an amplifier XII and a phase inverter XIII. When modulators with rotating shields are used this can be achieved by connecting two identical modulators on one shaft; the "base" voltage is created by feeding a dc voltage to the corresponding modulator. For a modulator which utilizes vibrating electrodes there is no necessity for a second modulator, and the "base" voltage is tapped off directly from the field circuit of the generator. The "base" voltage appears in the load circuit of the synchronous detector in phase-opposition and is thus eliminated, but the signal voltage appears in the load circuit in phase; the dc component of the signal voltage is registered by a dc galvanometer. The basic principle of the synchronous detector circuit, including the "base" voltage amplifier and the phase inverter, is depicted in Fig. 2. For the purpose of monitoring the gain of the microvoltmeter a calibrated oscillator XV has been provided for the circuit, with either an attenuator or (when calibrating the most sensitive ranges of the microvoltmeter) a high-precision variable resistor at its output.

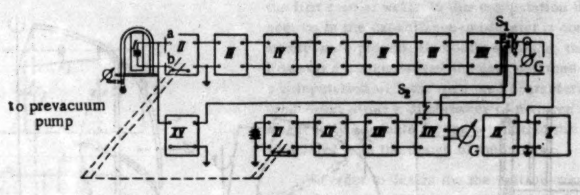
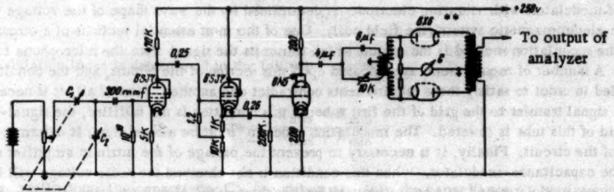


Fig. 1



* [6N5; E_p = 135 v, I_p = 110 ma, E_f = 6.3 v, I_f = 2.5 amp, P_p = 13 w, g_m = 6700 μmho, Γ_p = 460 r. - translator's note].

** [6X6 dual diode – E_p = 165 v, I_p = 16 ma, P.I.V. = 465 v, I_m = 50 ma, E_f = 6.3 v, I_f = 0.3 amp. – translator's note].

Fig. 2

The capacitance-modulator is a capacitor with a periodically varying capacitance, and it must satisfy the following requirements: the modulator must have the greatest possible variation in capacitance [here it is necessary, according to the concepts which are discussed below, that both the initial value of its capacitance and the capacitance from its electrodes to ground (chassis) be minimum]; the modulation frequency must lie below the investigated frequency range (in our case the modulation frequency is 24 cps); the modulator must have minimum intrinsic noise, and for this purpose a painstaking construction of its electrodes and contacts is necessary; its electrodes must be manufactured from the same piece of metal and must have some stable coating (silver, chrome).

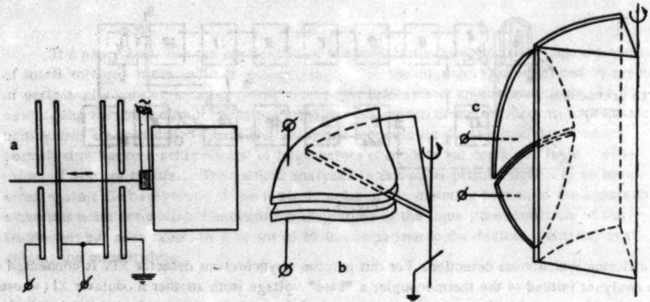


Fig. 3

We built three types of capacitance-modulators: 1) a modulator with vibrating electrodes (Fig. 3, a); 2) a modulator with a rotating grounded shield (Fig. 3, b); 3) a modulator with a rotating insulated shield (Fig. 3, c).

The function describing the variation of the capacitance of modulators with rotating shields is determined by the shape of the electrodes. In the instrument described here the electrodes are of such a shape that the area which is covered by the shield varies according to a sinusoidal law. The function describing the variation of the capacitance of modulators with vibrating electrodes is determined by the wave shape of the voltage which supplies the coil of the electromagnetic system (the field coil). One of the most essential sections of a circuit operating according to the modulation method is the circuit which transmits the signal from the microphone to the grid of the first tube. A number of requirements are imposed upon this section of the circuit, and the conditions which must be fulfilled in order to satisfy these requirements contradict one another. First of all, it is necessary to have the maximum signal transfer to the grid of the first tube; if this condition is not fulfilled, the signal-to-noise ratio at the grid of this tube is lowered. The modulation index m must be a maximum; it determines the effectiveness of the circuit. Finally, it is necessary to prevent the passage of the intrinsic amplifier noise to the electrode of the capacitance-modulator. When this condition is not observed the noise voltage will be modulated by the capacitance-modulator and will be transmitted further through the circuit together with the signal, thus distorting the signal. The same applies to the intrinsic noise of the capacitance-modulator.

The input circuit together with its equivalent circuit is depicted in Fig. 4, where C_1 is the capacitance-modulator, C_2 is the capacitance with respect to ground (which may be either a constant or variable capacitance of the output electrode of the modulator with respect to ground, or simply a constant capacitance), C_3 is an isolating capacitor which decreases the effect of the input capacitance of the tube upon the modulation index of the capacitance-modulator, C_4 is the dynamic input capacitance of the first tube of the preamplifier stage, Rg is the leakage resistance of the first tube.

Depending upon the nature of the test signal source, two cases may obtain :

a) the impedance of the source Z_g is much greater than the impedances Z_1 and Z_2; in this case the modulator must be connected in parallel with the test signal source;

b) the impedance Z_g is much less than the impedances Z_1 and Z_2; in this case the modulator must be connected in series with the test signal source.

For example, when measuring the noise of high-capacitance piezoelements made of Rochelle salt, the second case obtains. However, the computations made below for this case can easily be extended to apply to the first case as well. In this computation the process which goes on in the capacitance-modulator is considered to be a steady-state process; rigorously speaking, this is not true. However a comparison of the data obtained by means of such a computation with the measured characteristics of ready-made modulators shows a discrepancy of no more than 10%, and this is fully adequate for the solution of the practical problems associated with the design of modulators.

In order to determine the voltage-transfer coefficient of the input circuit let us cut the circuit shown in Fig. 4 along the line $\ell - \ell$ and apply the Thevenin theorem to its left-hand portion; if this is done the circuit will be transformed in the manner which is shown below in the same figure. If the sum of the impedances Z_3 and Z_4 is considerably greater than Z_1 and Z_2, then it is not necessary to take the reaction of the portion of the circuit to the right of the line $\ell - \ell$ into account. In that case we obtain the following simple expression for the voltage-transfer coefficient:

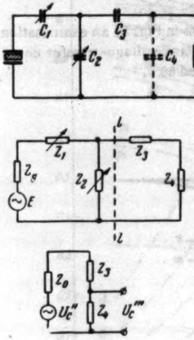


Fig. 4

$$K = \frac{C_1}{C_1 + C_2} \cdot \frac{C_3}{C_3 + C_4}$$

or, keeping in mind that for the specified circuit the capacitances C_3 and C_4 are constant,

$$K = \frac{C_1}{C_1 + C_2} .$$

The modulation index is determined in the following manner:

$$m = \frac{K_{\max} - K_{\min}}{K_{\max} + K_{\min}} .$$

If, as is usually done, the capacitance C_2 is constant and only the capacitance C_1 varies according to the law

$$C_1 = C_{1\min} + (C_{1\max} - C_{1\min})\left(\frac{1 + \sin \omega t}{2}\right),$$

where $C_{1\min}$ and $C_{1\max}$ are respectively the minimum and maximum values of the modulator capacitance, then, introducing the parameters

$$A = \frac{C_{1\,\min}}{C_{1\,\max}}, \quad x = \frac{C_{1\,\min}}{C_2},$$

which characterized the design of the modulator, we obtain the following expressions for the voltage-transfer coefficient and the modulation index after simple calculations:

$$K_{\max} = \frac{x}{x+A}, \quad K_{\min} = \frac{x}{x+1}, \quad m = \frac{1-A}{1+A+2x}.$$

On the basis of these relationships we have plotted a nomogram which is shown in Fig. 5; an examination of this nomogram shows that the most advantageous values for the modulation index and voltage-transfer coefficient (which can be attained for the best possible modulator design) do not exceed 50%.

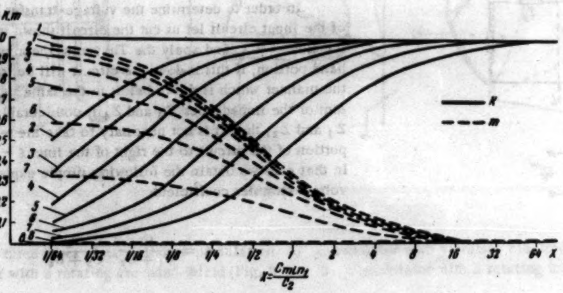


Fig. 5. Nomogram with the values of the parameter A: 1) 0; 2) 1/64; 3) 1/32; 4) 1/16; 5) 1/8; 6) 1/4; 7) 1/2; 8) 1.

Therefore we utilized a new "push-pull" input circuit whose principle consists of the simultaneous variation of the capacitances C_1 and C_2 in such a manner that these variations are out of phase by an amount equal to π. In that case,

$$C_1 = C_{1\,\min} + (C_{1\,\max} - C_{1\,\min})\left(\frac{1+\sin\omega t}{2}\right),$$

$$C_2 = C_{2\,\min} + (C_{2\,\max} - C_{2\,\min})\left(\frac{1-\sin\omega t}{2}\right).$$

If the capacitances C_1 and C_2 are identical (a symmetrical "push-pull" circuit), then we obtain the following expressions for the voltage-transfer coefficient and the modulation index:

$$K_{\max} = \frac{1}{1+A}, \quad K_{\min} = \frac{A}{1+A}, \quad m = \frac{1-A}{1+A}.$$

In this case it is theoretically possible to obtain the maximum values of K and m (values of the order of 90-95%), but practically speaking this case is not a real one because it is impossible to make the two modulators exactly identical. The capacitance of the modulator C_2 always includes both the capacitance of one of its electrodes as well as the capacitance of one of the electrodes of modulator C_1 with respect to ground (chassis); thus even when the capacitances between electrodes are equal, $C_{1\,\mathrm{min}}$ will always be less than $C_{2\,\mathrm{min}}$.

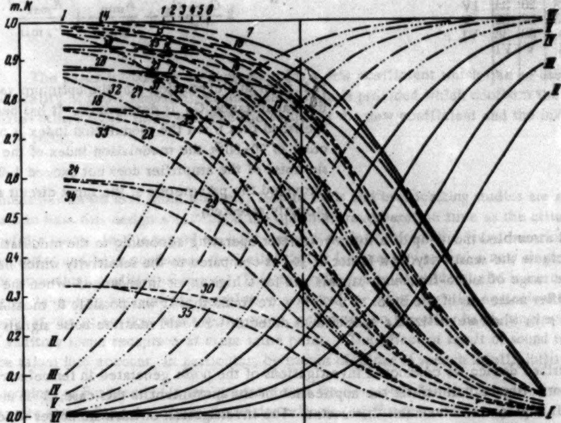


Fig. 6

When $C_1 \neq C_2$ (an asymmetrical "push-pull" circuit) we have (introducing the additional notation

$$A_1 = \frac{C_{1\,\mathrm{min}}}{C_{1\,\mathrm{max}}}, \quad A_2 = \frac{C_{2\,\mathrm{min}}}{C_{2\,\mathrm{max}}},$$

in order to characterize each of the modulators) the following expressions:

$$K_{\mathrm{max}} = \frac{x}{x + A_1}, \quad K_{\mathrm{min}} = \frac{A_2 x}{1 + A_2 x}, \quad m = \frac{1 - A_1 A_2}{2A_2 x + A_1 A_2 + 1}.$$

The nomogram for this case is depicted in Fig. 6. The designation of the curves is given in the table. It is not difficult to see that the region of the most advantageous values which can be obtained in this case for the modulation index and voltage-transfer coefficient is located between $x = 1/4$ and $x = 2$ when $1/2 > A_1 > 1/8$ and $1/2 > A_2 > 1/8$. Here the values of K and m oscillate between 80 and 90%. The proposed nomograms make it possible to make a preliminary determination of the values of the modulation index and the voltage-transfer coefficient for any modulator design by means of the calculated or measured values of the modulator capacitances.

TABLE

	m	A_2						
		0	1/16	1/8	1/4	1/2	1	K_{max}
A_1	0	1	7	13	19	25	31	I
	1/16	2	8	14	20	26	32	II
	1/8	3	9	15	21	27	33	III
	1/4	4	10	16	22	28	34	IV
	1/2	5	11	17	23	29	35	V
	1	6	12	18	24	30	36	VI
K_{min}		I	II	III	IV	V	VI	

By examining the input circuit shown in Fig. 4 from the opposite direction, it is not difficult to obtain the following expression for the modulation index produced in the intrinsic noise of the amplifier by the capacitance-modulator:

$$m_n = \frac{\dfrac{K_{min}}{C_{1\,min}} - \dfrac{K_{max}}{C_{1\,max}}}{2\dfrac{C_3 + C_4}{C_3 C_4} + \dfrac{K_{min}}{C_{1\,min}} + \dfrac{K_{max}}{C_{1\,max}}},$$

it is from this expression that the optimum value of the capacitance C_3 is determined. It has been established that when the modulation index is of the order of 70-80%, the modulation index of the intrinsic noise of the amplifier does not exceed 1-2% (provided the parameters of the input circuit are properly selected).

A fully designed and assembled mock-up of a microvoltmeter operating according to the modulation method permitted us to increase the sensitivity by a factor of 10 (as compared to the sensitivity which has been attained so far) over a wide range of audio-frequency signals and for a high input impedance. When the input level of the intrinsic amplifier noise was of the order of 5-7 μ v we found that it was possible to measure noise signals which began at 2.2 μ v; when we utilized a synchronous detector we could measure noise signals beginning from 0.7 μ v.

By means of the described device we conducted investigations of the noise generated in ferroelectric piezoelements and capacitors. Figure 1 illustrates the application of the instrument in this case; I is the piezoelement being tested, and it is placed in a vacuum thermostat. The investigation of intrinsic noises which are generated in ferroelectric elements showed that the level of these noises exceeds the calculated level for thermal noise; this indicates the presence of additional noise sources in ferroelectrics (in particular, noise generation as a consequence of the "Barkhausen effect" in a ferroelectric). The measurement of capacitor noise also yielded satisfactory agreement with the theoretical data which was obtained when we assumed that these noises were of a thermal nature. Thus it was established that the intrinsic noise generated in ferroelectric piezoelements cannot be considered purely thermal.

Among a number of possible applications of the method we have analyzed above, we should include the acoustical measurement of weak-signal levels and the calibration of microphones; it should be noted that this method can easily be applied in the ultrasonic frequency range as well.

Leningrad Shipbuilding Institute

Received April 25, 1955.

THE PROBLEM OF NEW COEFFICIENTS FOR EVALUATING THE QUALITY OF ROOM ACOUSTICS

E. E. Golikov

The paper introduces an expression for a new coefficient which can be used for evaluating the quality of room acoustics; experimental data is provided which confirms the presence of a single-valued relationship between the values of the new coefficient and the investigated point in the room.

Experiment has shown that when theaters, lecture halls and broadcasting studios are acoustically designed it is inadequate to base this design solely upon the optimum reverberation time as the criterion for the acoustical quality of the room. In this regard we have noted a deviation from the "reverberation" evaluations which are used in acoustics and have confirmed the belief that it is necessary to find a new single-valued criterion for evaluating the acoustical qualities of rooms. This new criterion must evidently be some coefficient whose value, on the one hand, takes into account the correlative effect of the most essential acoustical factors upon the reception of sound at one or another point in the room, and must, on the other hand, single-valuedly correspond to the effect of auditory sound reception at these same points. The effect or result of sound reception at these points must be taken into account, in particular, by means of values of speech intelligibility (articulation).

In the writings of Furduev [1] it was proposed that the so-called "acoustical ratio" could serve as such a criterion [i.e., the ratio of the density of the entire reflected (diffuse) sound energy at the point of reception to the density of the direct sound energy]. However it proved impossible to establish a single-valued relationship between the values of this coefficient and the intelligibility of speech.

Another criterion for evaluating the acoustical qualities of rooms — the so-called "reverberation interference factor" — was proposed by Sukharevskii and Rabinovich [2, 3]. This criterion is a coefficient which is known under the name of the "Strutt Criterion" and is refined by the elimination of two errors. However the errors in the "Strutt Criterion" mutually compensate each other to a considerable degree; therefore the Sukharevskii and Strutt criteria are essentially one and the same coefficient.

The reverberation interference factor (the Strutt criterion) is expressed in terms of the density ratio of the useful and "harmful" sound energy at the point of reception, since it is assumed that the resultant effect with respect to sound reception in the room must be determined by the relative effect of the useful and "harmful" components of the sound upon the hearing organ (the "harmful" sound energy is that which creates interference).

The density of the useful sound energy is considered to be the sum of the energy densities of the direct sound and that portion of the diffuse sound which consists of reflections that arise at the point of reception during the course of $^1/_{16}$ sec (50-60 milliseconds) and are received together with the direct sound.

The density of the "harmful" sound energy is considered to be the sum of the energy densities of the delayed reflections (those which arrive at the reception point later than $^1/_{16}$ sec) and of the extraneous sounds or noise.

In accordance with this, the expression for the reverberation interference factor can be written in the form

$$Q = \frac{E_p + E_d'}{E_d'' + E_h} \cdot * \qquad (1)$$

Here E_p is the density of the sound energy in the direct ray; this energy is determined according to the following function (when a nondirectional sound source is used):

$$E_p = \frac{P}{4 \cdot \pi \cdot c \cdot r^2}, \qquad (2)$$

where P is the acoustical power of the sound source, r is the distance of the point of reception from the sound source, c is the velocity of sound.

E'_d is the energy density of the useful sound reflections; this quantity is determined by the expression

$$E_d' = \frac{P}{V} \int\limits_{t=0}^{t=1/18} H(t) \cdot dt, \qquad (3)$$

where H(t) is the function which describes the decay of the reverberation sound in the given room after the source has ceased to emit sound, t is the time, V is the volume of the room.

E_d'' is the density of the sound energy contained in the "harmful" delayed reflections (the reverberation interference); this quantity is determined as

$$E_d'' = \frac{P}{V} \int\limits_{t=1/18}^{=\infty} H(t)\ dt; \qquad (4)$$

E_h is the density of the noise sound energy.

The statistical theory of room acoustics assumes that the decay of the reverberation sound can be represented by an exponential curve. According to this theory (i.e., according to the Eyring concept involved in the derivation of his formula for the reverberation time), if the sound source ceases to emit, then the following function results:

$$E_t = E_0 \cdot e^{\left(-\frac{\ln 10^6}{T}\right) \cdot t} = E_0 \cdot e^{-13.8\frac{t}{T}}, \qquad (5)$$

where E_t is the over-all reverberation energy in the room during a time t sec after the source has ceased to emit, E_0 is the over-all energy of reverberation sound in the room during the stable state that precedes the instant at which the source ceases to emit, T is the standard reverberation time. Then it follows that the function for the decay of the sound will be

---

* The name "reverberation interference factor" has been given to the quantity which is the inverse of the coefficient Q, but there will be no error in principle if, preferring to use the direct and not the inverse magnitude of the coefficient in our analysis, we shall apply this name to the coefficient Q.

$$H(t) = e^{-13.8\frac{t}{T}}. \tag{6}$$

In fact, due to the interference phenomena which, as a rule occur in almost all rooms, the attenuation curve for the sound will not have the form of a smooth exponential curve but will consist of a great number of nonuniformities which, as it were, are disposed around the basic attenuation curve.

However, if we allow for a certain amount of error, then it is possible to assume that in accordance with the derivations of the statistical theory the attenuation function H(t) for the sound in the room is exponential. Here

$$E'_d = \frac{P}{V}\int_0^{1/16} e^{-13.8\frac{t}{T}}\,dt = \frac{PT}{13.8\,V}\left[1 - e^{-\frac{13.8}{16\cdot T}}\right], \tag{7}$$

$$E''_d = \frac{P}{V}\int_{1/16}^{\infty} e^{-13.8\frac{t}{T}}\,dt = \frac{PT}{13.8\,V}\cdot e^{-\frac{13.8}{16\cdot T}} \tag{8}$$

For a directional source (receiver) of sound the directionality can be taken into account in the expression for the coefficient Q by means of introducing the coefficient of axial concentration θ_0 and the polar diagram factor R (α, β) (when the angular coordinates for the direction from the sound source to the point of reception are α and β).

Foreign authors [4] propose that the so-called "liveness factor" of rooms should be taken as the single-valued criterion for judging acoustical quality.

Although this proposal is the later one chronologically, its authors permit the same very substantial inaccuracy to appear in their determination of the proposed criterion as was permitted in the determination of the acoustical ratio. This inaccuracy consists of the absence of a differentiated consideration of the reflected sound rays according to the temporal mode of their arrival at the reception point.

The expression for the liveness factor is again given in the form of a ratio between the "harmful" and useful components of the sound. Here the "harmful" component of the sound is taken into account by a time integral of the energy density of the entire reverberation sound, and the useful sound component is represented by the unintegrated magnitude of the direct sound energy density. In accordance with this the liveness factor is expressed in the form

$$L = \frac{\int_{t_1}^{t_2} E_d\cdot dt}{E_p}, \tag{9}$$

where t_1 and t_2 are the time limits for the existence of reflected waves (from a sound pulse) in the room, E_d is the energy density of the entire reverberation sound.

The expression for the liveness factor (9) can be reduced to the form

$$L = \frac{4\cdot\pi\cdot c\cdot T^3\cdot r^2}{(13.8)^3\cdot V}. \tag{10}$$

after certain transformations.

It should be noted that several other coefficients for the qualitative evaluation of room acoustics [5] have been proposed. However, in the majority of cases they are proposed not as single-valued criteria of the acoustical quality of rooms but as certain characteristic quantities which supplement the reverberation criteria.

Thus, of all the variants proposed as a single-valued criterion for the articulation quality of rooms, the Sukharevskii-Strutt Criterion (1) should be considered the most correct and accurate. Nevertheless, the values of this coefficient and the magnitudes of the actual articulation quality at one or another point in a room can be proven to correspond only approximately. We are convinced of this by calculating the reverberation interference factor for the conditions of one or another set of articulation measurements and then comparing these values with the magnitudes obtained for the articulation.

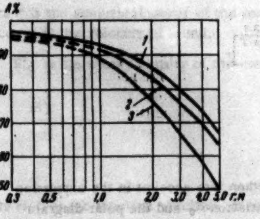


Fig. 1. 1) for a reverberation time of T = 1.2 sec, 2) for T = 2.2 sec, 3) for T = 4 sec.

As an example, let us compare the values of the reverberation interference factor with the magnitudes of the articulation under the conditions which obtained for the well-known articulation measurements made by Steinberg (the results of these measurements are shown in Fig. 1 by graphical plots of articulation vs distance between the speaker and the reception point for various reverberation times [6]).

The conditions under which these measurements were made were the following: the volume of the room was equal to 255 m^3; the reverberation time had values of 1.2, 2.2 and 4 seconds; the distance between the speaker and the reception point was varied within the limits of 0.3-5 m. The average acoustical speech power of the speaker was of the order of 10 microwatts. Measures were taken to eliminate noise, since the purpose of the experiment consisted of determining solely the effect of the reverberation interference upon the articulation.

Figure 1 shows graphs for the variation of syllabic articulation A as a function of the distance $\underline{r}$ from the speaker for various reverberation times (these graphs were plotted from the data obtained for these experiments).

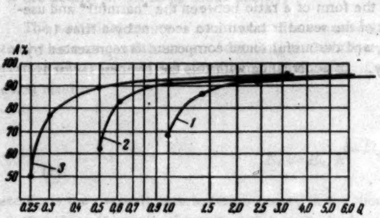


Fig. 2. 1) for T = 1.2 sec, 2) for T = 2.2 sec, 3) for T = 4 sec.

We calculated the values of the reverberation interference factor Q [these values were determined in accordance with Equations (1), (2), (7) and (8) and which corresponded to the above experimental conditions, for these same values of reverberation time]. On the basis of these calculations and the data shown in Fig. 1, we plotted graphs A = f (Q) for the corresponding reverberation times (these graphs are shown in Fig. 2).

It can easily be seen that considerably different values of the reverberation interference factor factor correspond to identical articulation values; we have three curves which do not coincide with one another. In an analogous manner we plotted the graphs A = f (L) for the conditions which obtained for the very same articulation measurements of Steinberg (Fig. 3) ; here we again see three non-coincident curves; if there were a single-valued relationship between the articulation and the coefficient L, then all three curves would be superimposed into one line. As we might expect, the curves $A = f(L)$ have a greater divergence than the curves $A = f(Q)$.*

TABLE 1

r, м / T, sec	0.3	1	2	5
1.2	16.7	2.5	1.4	1.0
2.2	6.65	1.03	0.618	0.510
4	3.07	0.495	0.305	0.250

Such a verification of the presence of a single-valued relationship between the reverberation interference factor (or the liveness factor) and the magnitudes of the articulation yields an analogous result when performed on the basis of other articulation measurements. A single-valued relationship is not observed. Thus the proposed criteria for the articulation quality of rooms require further refinement.

A certain possibility for refining the expression for the single-valued criterion of the articulation quality of rooms was noted by us in analyzing the calculated values of Q (Table 1). The values for the reverberation interference factor which correspond to approximately one and the same magnitudes of articulation (but which are determined for different values of reverberation time) proved to be inversely proportional to the reverberation time. The values of Q which were calculated for a reverberation time of T = 1.2 sec were approximately twice as large as the values of Q which were obtained for T = 2.2 sec and were four times as large as values of Q which corresponded to a reverberation time of T = 4 sec.

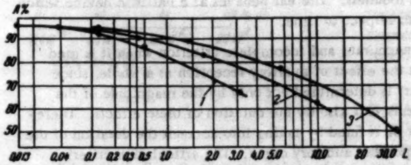


Fig. 3. 1) for T = 1.2 sec, 2) for T = 2.2 sec, 3) for T = 4 sec.

Therefore it can be assumed that if an additional factor in the form of the reverberation time T is introduced into the expression for the reverberation interference factor (1), then such an expression will determine the values of a certain new coefficient $Q' = QT$, and these values will more accurately correspond to the magnitudes of the articulation.

Figure 4 shows graphs of the syllabic articulation A_1 as a function of this new coefficient Q' for the same conditions as obtained for the articulation measurements of Steinberg and at the same points as were used in the preceding cases.

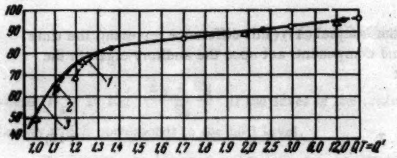


Fig. 4. 1) for T = 1.2 sec, 2) for T = 2.2 sec, 3) for T = 4 sec.

It can easily be seen that all three curves of $A = f(Q')$ which correspond to the three values of reverberation time coincide amazingly well and form one line; this indicates the presence of an extremely accurate single-valued relationship between the values of Q' and A. Thus the factor in the form of the reverberation time T which was introduced into the expression for the reverberation interference factor is actually a factor which makes the expression for the single-valued criterion of the articulation quality of rooms more accurate.

---

* In evaluating the degree of divergence for the curves $A = f(Q)$ (Fig. 2) and the curves $A = f(L)$ (Fig. 3) it should be taken into account that the logarithmic scale of the axis of abscissas in Fig. 2 accentuates the divergence of the curves, while in Fig. 3 it conceals the divergence.

We postulate that this is not accidental, and we shall show that this factor is actually based upon the physical and physiological phenomena which are associated with sound reception.

An important place in the explanation of the special features of auditory reception must be accorded to the discovery of the integral laws of auditory reception by S. Ia. Lifshits.

The integral loudness law is expressed in the form

$$G = \log \int_{t_1}^{t_2} I \cdot dt = \log (I \cdot t),$$

where G is the loudness, I is the sound intensity and t is the time for which the sound exists. The loudness G (for the same sound intensity) increases with the time for which the sound acts. The second integral law which governs the duration of the sound is also expressed in the form of a time integral:

$$D = \int_{t_1}^{t_2} \log I \cdot dt = \log I \cdot t.$$

The apparent duration D of a pulse depends upon its loudness. The ear behaves as a ballistic device which sums (integrates) the effect of auditory sound reception with respect to time.

Thus the reverberation interference factor (1) is an inaccurate and incomplete criterion when it is used as the criterion for articulation quality. It does not reflect the effect of auditory reception as a whole, since auditory reception (due to the ballistic properties of the ear) is determined not only by the magnitude of the relative effects of the useful and "harmful" sound components, but also by the duration of these effects. Therefore the criterion for the articulation quality of rooms must be refined by taking into account the duration of the relative effects of the useful and "harmful" sound energy upon the auditory organ. The articulation criterion must be not only the ratio of the densities of the useful and "harmful" sound energies, but also the time integral of this ratio, i.e., it must be of the form:

$$Q' = \int_{t_1}^{t_2} \frac{E_p + E_d'}{E_d'' + E_h} dt. \qquad (11)$$

Since the difference between the integration limits of the Integral (11) (this difference represents the time for which the relative effects of the useful and "harmful" sound components act upon the auditory organ in the room) is the reverberation time of the room T, it follows that

$$Q' = \frac{E_p + E_d'}{E_d'' + E_h} \cdot T. \qquad (12)$$

When the ballistic properties of the ear are taken into account we obtain the same time factor in the form of the reverberation time T (the necessity of introducing such a factor is indicated because of the regularity of the numerical values of the reverberation interference factor; this regularity has been discussed above). Therefore the introduction of this corrective factor must be considered to be completely justified and necessary for refining the criterion of the articulation quality of rooms.

The analysis of the expression for the Sukharevskii-Strutt Criterion (1) [this expression can be written in expanded form by taking into account Equations (2), (7) and (8)] shows that when there is no noise (i.e., for the condition $E_h = 0$) the magnitude of the acoustic power proves to be a common factor in the numerator and denominator of Expression (1) and can be eliminated by reduction. This means that the expression for the Sukharevskii-Strutt Criterion yields the same values, regardless of the acoustic power of the sound source and the absolute values of sound energy density. In other words, the expression for the Sukharevskii-Strutt Criterion does not take into account an important acoustical factor – the loudness; this is still another substantial inaccuracy of this criterion.

This inaccuracy is evidently also contained in the expression for the new coefficient proposed for the qualitative evaluation of room acoustics (12), which is derived from Expression (1). In accordance with this expression, the values of the criterion Q' which are determined by Expression (12) for any points in the room may be viewed as corresponding solely to the condition of optimum sound level at the observation points. If, however, the level of the useful sound is less than the optimum at the examined point in the room, then the value of the criterion must also be less and must correspond to a lower articulation quality at lower levels of useful sound.

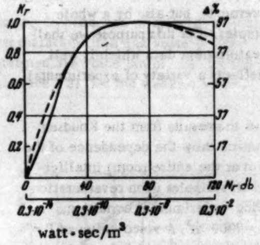


Fig. 5

In order to ensure that the criterion of the articulation quality of rooms will also take into account the phenomenon of a lowered articulation that is caused by a lowered sound level, it is evidently necessary to introduce still another additional factor. The values of this factor must be determined as a function of the useful sound level at the reception point, and at the same time they must reflect the character of the dependence of articulation upon the sound level.

If we introduce the quantity K_r for this factor and take into account the known graphical dependence of articulation upon sound level (this dependence was experimentally found by Fletcher and Steinberg and is depicted by the solid curve in Fig. 5), then the values of K_r can be determined (as a function of one or another sound level at the examined point) as ordinates of the graph which corresponds to these levels; these ordinates are taken relative to the ordinate which corresponds to the optimum level. Here it is possible to obtain a graphical dependence for the values of K_r upon the sound level (this dependence is also shown in Fig. 5 and is depicted by the dashed curve). This curve can be described by the following equation with sufficient accuracy for practical calculations:

$$K_r = 0.0388 \cdot N_r e^{-\frac{N_r}{70}}, \tag{13}$$

where $N_r = 10 \log \dfrac{E_p + E'_d}{E_0}$ is the level of the useful sound at the reception point; E_0 is the volume energy density which corresponds to the null level.

Thus the expression proposed for the new coefficient which can be applied in the qualitative evaluation of room acoustics will have the form:

$$Q'' = \frac{(E_p + E'_d) \cdot T \cdot K_r^*}{E''_d + E_h}. \tag{14}$$

---

[*] The Expression (12) written earlier should be viewed as a particular form of the expression for the coefficient Q' when $K_r = 1$.

it will contain the corrective factors T and K_r, and will take into account the integral properties of the ear as well as the loudness factor.

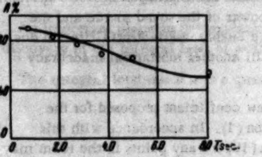


Fig. 6. The dependence of the percentage syllabic articulation upon the reverberation time of a room with a volume of V = = 7000 m^2 (according to the Knudsen method).

For the conditions which obtained for the Steinberg articulation measurements (these measurements were made in a room of small volume where V = 225 m^3 and the sound levels were optimum) K_r was equal to unity and the values of the coefficient Q*, as well as the magnitudes of the articulation which corresponded to them, remained the same as those shown in the graphs of Fig. 4.

We shall show that the presence of an accurate single-valued relationship between the compared values of the coefficient Q* and the articulation A is verified not only by the example of the Steinberg articulation measurements, but also by a whole series of other examples. For this purpose we shall use articulation measurement data which is well known and which reflects a variety of experimental conditions.

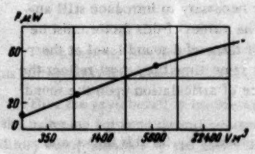


Fig. 7. Dependence of the probable speech power in microwatts upon the volume of the room.

Figure 6 shows the results from the Knudsen measurements [7] which show the dependence of the average (averaged over the entire room) intelligibility of actual speech syllables upon reverberation time for the following experimental conditions: a room volume of V = 7000 m^3, a voice power of P = = 0.46 · 10^{-4} watts for the speaker (this is established from the known graphical dependence shown in Fig. 7), a noise which has been weakened to a level which permits it to be ignored when the speech is directly received binaurally and when the distance between the observation point and the speaker is r ≃ 12 m. The results of the articulation measurements, averaged over the entire room, can be viewed as the results which are obtained at an average point in the room; for a given room volume, a usual relationship between the linear dimensions of the room (3 : 2 : 1, or 5 : 3 : 2; the length of the hall was ℓ = 30-31 m) and a speaker position which is typical for articulation measurements (in the direction of the length of the room and at a distance of several meters from a wall), the selected value of r can be considered to be a value characterizing precisely the average point.

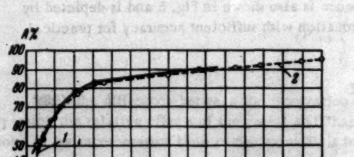


Fig. 8. 1) For the articulation conditions which obtained for the Knudsen measurements, 2) for the articulation conditions which obtained for the Steinberg measurements.

Under the conditions specified above for the Knudsen articulation measurements and for several values of reverberation time (precisely those values for which individual experiments were conducted; the results of these experiments are shown by the points in Fig. 6) we calculated the values of the coefficient Q*. In Fig. 8 they are compared with the magnitudes of A by means of a graph A = f(Q*).

A comparison between the graphs of A = f(Q*) obtained for the conditions under which the Knudsen and Steinberg articulation measurements were made show (in the same figure) their identity (the curves coincide), and this demonstrates the single-valued relationship between the coefficient Q" and the magnitudes of the articulation.

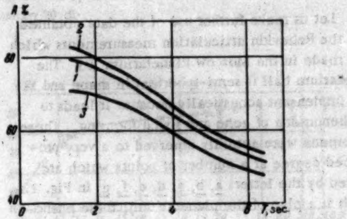


Fig. 9. Dependence of syllabic articulation upon reverberation time and voice power; 1) for an intermediate voice power, 2) for the most powerful voice, 3) for the weakest voice.

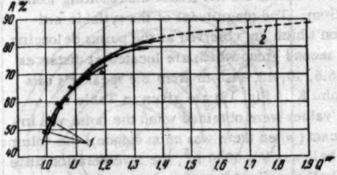


Fig. 10. 1) For the articulation conditions which obtained for the Knudsen measurements, 2) for the conditions which obtained for the Steinberg measurements.

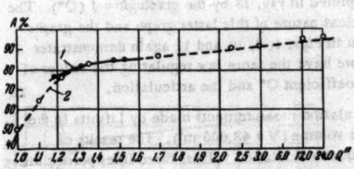


Fig. 11. 1) For the conditions under which the Engel articulation data were obtained, 2) for the conditions which obtained for the Steinberg articulation measurements.

The family of curves shown in Fig. 9 represents the data from the Knudsen articulation measurements; this data establishes the dependence of the articulation values (averaged over the entire room) upon the reverberation time and the power of the natural voice of the speaker. This data corresponds to different experimental conditions [7].

The volume of the rectangular room was V = = 11,200 m^3. In accordance with the graph shown in Fig. 7, the average speech power of the orator in a room of the specified volume was equal to 54 microwatts. Respectively, the values of power for the strongest and weakest voices were 132 and 9 microwatts, since (according to the Knudsen data) the values of power for the strong, intermediate and weak voices in his experiments were related by the ratios 66.2 : 27.4 : 4.5. The noise level was insignificant, and the sound reception was direct and binaural. Finally, on the basis of the concepts introduced above, it is not difficult to establish that the distance of the average point in the room from the speaker was r = = 16 meters; the average results of the Knudsen measurements can be referred to this point.

Under the above experimental conditions we calculated the values of the coefficient Q" which corresponded to three different values of the acoustical voice power of the speaker; the measurements were repeated for these values. In addition, these values were calculated as a function of reverberation time (the reverberation time is the second variable factor in the measurements being examined) and were compared with the magnitude of the observed articulation.

Figure 10 shows the graphical dependencies A = f (Q") for this case as well as for the Steinberg measurements; these graphs again show the single-valued nature of the relationship between the coefficient Q" and the articulation. Figure 11 shows analogous graphs which were plotted on the basis of data taken from the book by Engel [8] which gives values for the percentage syllabic articulation that is observed (on the average) for a hall with a volume of 2800 m^3 when the hall is filled to various degrees, as well as the values of reverberation time which correspond to these conditions (here, the fact that noise was present in the hall was taken into consideration; the values of the noise level corresponded to the various degrees to which the hall was filled and were equal to 40, 35, 30 and 20 db).*

The graphs $A = f(Q^*)$ as in the preceding cases, coincide with the analogous graphical dependency which was obtained for the conditions that were specified for the articulation measurements examined earlier.

Let us make further use of the data obtained from the Rzhevkin articulation measurements which were made in the Moscow Planetarium [9]. The planetarium hall is semi-spherical in shape and is most unpleasant acoustically because it leads to the phenomena of echo and sound focusing. These phenomena were actually observed to a very pronounced degree at a number of points which are denoted by the letters a, b, c, d, e, f, g in Fig. 12 (which is a plan of the planetarium); these phenomena are concentrated in the regions which are called "anomalous" zones by Rzhevkin.

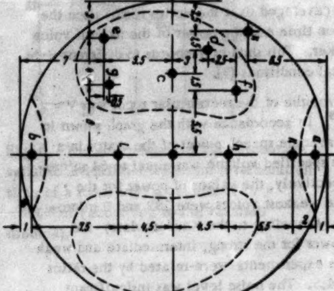


Fig. 12

At the points belonging to the other group – h, i, k and l – (which are located outside of the anomalous zones enclosed by the dashed line) echo and the concentration of sound rays were definitely not observed (i.e., the acoustic defects in the shape of the hall did not affect the resulting sound reception). The magnitudes of the syllabic articulation which are observed at the points belonging to the second group which are located at distances of $r = 5.5$, 10, 12, 14.5 m from the speaker's dais (the point K in Fig. 12) are given in Table 2. These values were obtained when the noise was insignificant (when there was no audience); the volume of the hall was $V = 5000\ m^3$, the reverberation time of the hall was $T = 1.8$ sec, natural speech was transmitted, and the probable speech power was $P = 0.43 \cdot 10^{-4}$ watts for the given volume. The values of the coefficient Q^* which correspond to the specified conditions and points are given in the same table, and the result of a comparison between the corresponding values of Q^* and the articulation is depicted in Fig. 13 by the graph $A = f(Q^*)$. The identical nature of this latter graph and the graphs shown in Figs. 4, 8, 10 and 11 again demonstrates that we have the same law regulating the values of the coefficient Q^* and the articulation.

TABLE 2

Points	h	i	ℓ	k
r, м	5.5	10	12	14.5
Q'	1.91	1.32	1.26	1.21
A in percent	88	82	78	74

Finally, let us make use of the data obtained from the articulation measurements made by Lifshits in the Miussk Cathedral in Moscow [10] which is a hall with a very large volume ($V = 43{,}000\ m^3$). The results of these measurements are shown in Fig. 14 by the closed lines for equal articulation values; these lines permit the values of articulation to be read for any point in the hall. The picture of the articulation field depicted in this figure is obtained for the case where the speaker is located at point O. In addition, the following experimental conditions are known: the reverberation time was T = 2 sec ; according to Fig. 7 the probable acoustical speech power of the speaker was equal to $P = 0.63 \cdot 10^{-4}$ watts for the given volume of the hall ; the noise level was equal to 30 db.

---

* The average point in the room was taken as the observation point when calculating the values of the coefficient; this point was assumed to be a distance $r = 9.5$ m away from the speaker, since for the specified volume and the usual relationship between the linear dimensions of the hall the length of the hall was 22 m.

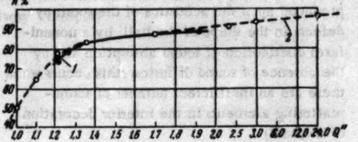


Fig. 13. 1) For the conditions which obtained for the Rzhevkin articulation measurements, 2) for the conditions which obtained for the Steinberg measurements.

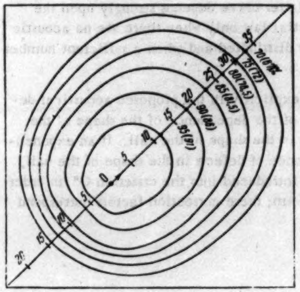


Fig. 14

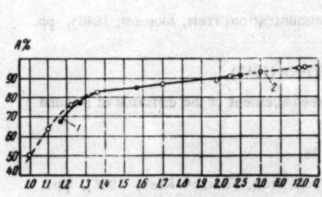


Fig. 15. 1) For the conditions which obtained for the Lifshits articulation measurements, 2) for the conditions which obtained for the Steinberg measurements.

Figure 15 shows the graph $A = f(Q^{"})$ corresponding to this data; this graph shows that a single-valued relationship between the values of the coefficient $Q^{"}$ and the articulation was also observed under the conditions which obtained in the Miussk Cathedral (this is the same relationship which was observed for articulation measurements which were made in halls with a smaller volume).

Since the observed single-valued relationship between the values of the coefficient $Q^{"}$ and the magnitudes of the articulation is always expressed in terms of the same curve on the graph $A = f(Q^{"})$ this curve can evidently be considered as the scale for the single-valued relationship between the values of the coefficient $Q^{"}$ and the values of articulation . This curve, plotted according to corresponding values of $Q^{"}$ and A for all of the examined examples of articulation measurements, is shown in Fig. 16.

After having made the above refinements in the expression for the single-valued criterion of the acoustic quality of rooms and having reduced it to the form of Expression (14), we see that this latter expression takes into account the effect of all the basic acoustic factors upon the acoustic results. The expression for the criterion takes the following factors into account: the noise and reverberation interference, the relationship between the useful and "harmful" sound components, the absolute values of the sound level or the loudness factor, the directionality of the source (receiver) of the sound and the integral laws governing auditory reception. In addition, the criterion takes into account the presence not only of reverberation but also of direct sound. The former is taken into account by assuming that it arrives at the reception point in a time-differentiated manner and by taking into consideration the particular features of the subjective reception of signals which are close to one another in time.

When the basic acoustic factors are thus taken into rather complete consideration in the expression for the coefficient $Q^{"}$, the observed single-valued relationship between the values of this coefficient and the values of articulation is assumed to be regular. Therefore the expression for the proposed new coefficient of qualitative evaluation of room acoustics and the graphical scale for the single-valued relationship between the values of this coefficient and the magnitudes of the articulation can be proposed in the capacity of design functions for the proper acoustic design of halls.

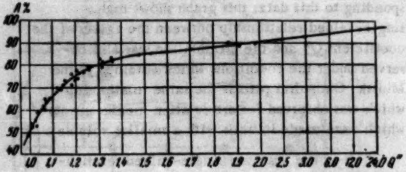


Fig. 16

In the expression for the new coefficient of qualitative evaluation of room acoustics the only factors which are not taken into consideration in an explicit form are the known effects produced upon the acoustics of the room by defects in the shape of the hall, by a nonuniform distribution of sound absorption and by the absence of sound diffusion (this occurs when there are an insufficient number of sound-scattering elements in the interior decoration of the hall).

However, the shape factor of the hall is taken into account in the expression for the coefficient Q'' in a nonexplicit manner. In fact, the quantities which represent the energy densities of the useful and "harmful" reflections (E_d and E'') and which are included in the expression for the coefficient are determined as a function of the shape of the attenuation curve H (t) for the reverberation sound. This latter curve depends strongly upon the nature of the interior decoration of the hall and approaches an exponential law only when there are no acoustic defects in the shape of the hall, when the sound absorption is uniformly distributed and when a sufficient number of sound-scattering elements are present.

Therefore if the shape factor of halls is to be taken into account explicitly in the proposed acoustical design functions, it is necessary to pay attention to the degree and nature of the dependence of the shape of the reverberation attenuation curve H (t) upon one or another set of defects in the shape of the hall. If an exponential attenuation curve for the reverberation sound corresponds to an absence of defects in the shape of the hall, then when defects are present one or another set of corrections must be introduced into the criterion Q'' in order to correct for the deviation of the function H (t) from the exponential form; these correction factors correspond to the existing defects in the shape of the hall.

LITERATURE CITED

[1] V. V. Furduev, "A method for the acoustic design of auditoriums which are equipped with loudspeakers," Doklady Akad. Nauk SSSR 1937, No. 15, 313-316.

[2] Iu. M. Sukharevskii, " The effect of acoustic conditions in a room upon the intelligibility of reproduction"(Chapter from a Dissertation) (Moscow Power Institute, 1940).

[3] I. G. Dreizen, A Course in Electro-Acoustics, Part II (Communications Press, Moscow, 1940), pp. 182-184.

[4] I. Maxfield and W. Albersheim, J. Acoust. Soc. Am. 19, 71-79 (1947).

[5] V. V. Furduev, "Survey of the methods of evaluation and measurement of the diffusion of a sound field," Acoust. J. 1, 4, 229-313 (1955).

[6] I. Steinberg, Electrical Engineers Handbook, Acoustics, 1938.

[7] V. O. Knudsen, Architectural Acoustics (Ukrainian Sci.-Tech. Press, Kharkov-Kiev, 1936), pp. 342-347.

[8] I. Engl, Raum- und Bauakustik, (Leipzig, 1939), pp. 250-252.

[9] S. N. Rzhevkin, "Investigation of articulation and speech intelligibility in the Moscow Planetarium," J. Tech. Phys. (USSR) 5, 1303-1308 (1935).

[10] S. Ia. Lifshits, "Acoustics of large auditoriums," J. Tech. Phys. (USSR) 3, 1062-1066 (1933).

Tomsk Electromechanical Institute of
Railway Transport

Received December 1, 1955.

CALCULATION OF THE RADIATION RESISTANCE OF SEVERAL TYPES OF DISTRIBUTED RADIATOR SYSTEMS

M. I. Karnovskii

The paper determines the resistive component of the radiation impedance of several distributed systems of coherent radiators. Groups of spherical radiators are examined.

In calculating distributed systems of sound radiation the interaction between loudspeakers is usually not taken into account (this interaction leads to a variation in the radiation impedance of the individual loudspeakers); the radiation impedance of each source in the group is assumed to be the same as that of a single (isolated) source. However, when the sources are placed at distances from one another which are smaller than or comparable to a wavelength,, the radiation impedance of a source in the group may differ substantially from the radiation impedance of an isolated source.

This paper determines the resistive component of the mechanical radiation impedance of various groups of spherical radiators operating in a regime of a constant wave velocity while radiating in-phase and phase opposition sinusoidal waves (the case where coherent noise is radiated is also analyzed). The radius of the spherical radiators is assumed to be small in comparison to a wavelength.

1. A Group Consisting of Two Primitive Spherical Radiators

Let us examine a group which consists of two identical primitive [1] spherical radiators which are located at a distance d from one another. The radiators are positioned in such a way that the direction of the maximum radiation from each isolated radiator coincides with the direction $\theta = 0$ (Fig. 1).

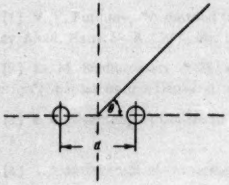


Fig. 1

The power radiated by the system will be

$$W = 2\pi r^2 \int_0^\pi I(\theta) \sin\theta \, d\theta.$$

where $I(\theta)$ is the sound intensity at a remote point at an angle θ and at a distance r from the system. The expression for W can be reduced to the form

$$W = 8\pi r^2 I_0 \int_0^\pi S_1^2(\theta)\, S_{gr}^2(\theta) \sin\theta \, d\theta,$$

where I_0 is the sound intensity of an isolated source for $\theta = 0$, $S_1(\theta)$ is the normalized directionality coefficient for an isolated source, $S_{gr}(\theta)$ is the normalized directionality coefficient for an analogous group of point sources. For in-phase (or phase-opposition) operation of the radiators the power radiated by each radiator of the group will be

$$W_{gr} = \frac{W}{2} = 4\pi r^2 I_0 \int_0^\pi S_1^2(\theta) S_{gr}^2(\theta) \sin\theta \, d\theta. \tag{1}$$

For an n-th-order primitive radiator

$$S_1(\theta) = P_n(\cos\theta), \tag{2}$$

where $P_n(\cos\theta)$ is a Legendre polynomial of the n-th order. Therefore the power radiated by a primitive radiator in the group is

$$W_{n\,gr} = 4\pi r^2 I_0 \int_0^\pi P_n^2(\cos\theta) S_{gr}^2(\theta) \sin\theta \, d\theta, \tag{3}$$

whereas the power of an isolated primitive radiator of the n-th order is

$$W_n = 2\pi r^2 I_0 \int_0^\pi P_n^2(\cos\theta) \sin\theta \, d\theta = \frac{4\pi r^2 I_0}{2n+1}. \tag{4}$$

Denoting the resistive component of the mechanical radiation impedance of a primitive radiator in the group and of an isolated radiator by $R_{n\,gr}$ and R_n respectively, we obtain

$$\frac{R_{n\,gr}}{R_n} = \frac{W_{n\,gr}}{W_n} = (2n+1)\int_0^\pi P_n^2(\cos\theta) S_{gr}^2(\theta) \sin\theta \, d\theta. \tag{5}$$

When the radiators operate in phase

$$S_{gr}(\theta) = \cos(a\cos\theta),$$

where $a = kd/2$, $k = \omega/c$ (c is the velocity of sound), and we obtain

$$\frac{R_{n\,gr}}{R_n} = 1 + \varepsilon_n, \tag{6}$$

where

$$\varepsilon_n = \frac{2n+1}{2}\int_{-1}^{1} P_n^2(y) \cos 2ay \, dy. \tag{7}$$

When the radiators operate in phase-opposition, $R_{n\,gr}/R_n$ reduces to the form

$$\frac{R_{n\,gr}}{R_n} = 1 - \varepsilon_n. \tag{8}$$

Assuming that n = 0 and $P_0(y) = 1$ in Equation (7), we obtain the well-known relationship (Fig. 2)

$$\varepsilon_0 = \frac{\sin kd}{kd}. \tag{9}$$

For a group of dipoles n = 1, $P_1(y) = y$, and therefore

$$\varepsilon_1 = 3\frac{\sin kd}{kd} + 6\frac{\cos kd}{(kd)^2} - 6\frac{\sin kd}{(kd)^3} \tag{10}$$

The values of ϵ_1 are given in Fig. 2.

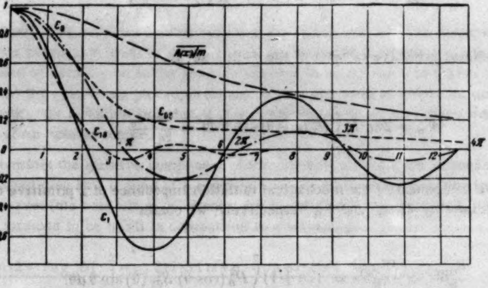


Fig. 2. The following quantities are plotted along the axis of abscissas: kd for the curves ϵ_0 and ϵ_1; k_0p for the curves ϵ_{0s} and ϵ_{1s}; kl for the curve $\overline{A(x)}/m$.

Let us now determine the values of ϵ_{ns} under the condition that the sources must radiate identical complex signals which are stationary in the probability sense. The nature of the signal shall be evaluated by means of the magnitude of the autocorrelation coefficient (a normalized function) $\varphi(\tau)$.

The directionality coefficients of a system consisting of two point sources will be [2]

$$S_{gr}(\theta) = \sqrt{\frac{1}{2}[1 \pm \varphi(\tau)]}, \tag{11}$$

where $\tau = \frac{d}{c}\cos\theta$.

Substituting the value of $S_{gr}(\theta)$ into Equation (5), we obtain

$$R_{ngr}/R_n = 1 \pm \varepsilon_{n_s}, \tag{12}$$

where

$$\varepsilon_{n\,s} = \frac{2n+1}{2}\int_0^\pi P_n^2(\cos\theta)\,\varphi\left(\frac{d}{c}\cos\theta\right)\sin\theta\,d\theta. \tag{13}$$

Let us determine $\epsilon_{0\,s}$ for n = 0 and for conditions under which a band of white noise with a width of $\Delta\omega$ and a mean frequency of $\omega_0 = k_0 c$ is radiated. In that case

$$\varphi(\tau) = \frac{\sin(\Delta\omega\tau/2)}{\Delta\omega\tau/2)}\cdot\cos\omega_0\tau = \frac{\sin(2a_0p\cos\theta)}{2a_0p\cos\theta}\cdot\cos(2a_0\cos\theta),$$

where

$$p = \frac{\Delta\omega}{2\omega_0}, \quad a_0 = \frac{k_0 d}{2},$$

Therefore,

$$\varepsilon_{0\,s} = \frac{1}{2}\int_{-1}^{1}\frac{\sin 2a_0py}{2a_0py}\cos 2a_0y\,dy.$$

Integrating this expression, we obtain

$$\varepsilon_{0s} = \frac{1}{2k_0dp}\{\mathrm{Si}\,[k_0d(1+p)] - \mathrm{Si}\,[k_0d(1-p)]\}. \tag{14}$$

The values of $\epsilon_{0\,s}$ for p = 1

$$\varepsilon_{0\,s} = \frac{\mathrm{Si}\,2k_0d}{2k_0d} \tag{15}$$

are given in Fig. 2. From Expression (14) it follows that for small values of $k_0 d$, $\epsilon_{0\,s} \approx 1$ for any $p (0 \leq p \leq 1)$.

Expression (14) can also be obtained on the basis of the following concepts. When a band of white noise is radiated it follows from Equation (9) that a definite value of ϵ_0 corresponds to each value of $k = \omega/c$. Averaging ϵ_0 over the specified range of frequencies, we obtain $\epsilon_{0\,s}$ (i.e.,

$$\bar{\varepsilon}_0 = \frac{1}{\Delta k}\int_{k_0-\frac{\Delta k}{2}}^{k_0+\frac{\Delta k}{2}}\frac{\sin kd}{kd}\,dk = \varepsilon_{0s}$$

Analogously we can obtain the value of $\epsilon_{1\,s}$ when a band of white noise is radiated by a system consisting of two dipoles. In this case

$$\varepsilon_{1s} = \frac{1}{\Delta k}\int_{k_0-\frac{\Delta k}{2}}^{k_0+\frac{\Delta k}{2}}\varepsilon_1\,dk.$$

Taking (10) into account, we find

$$\epsilon_{1s} = \frac{3}{2k_0dp}\left\{\frac{\sin[k_0d(1+p)]}{[k_0d(1+p)]^2} - \frac{\sin[k_0d(1-p)]}{[k_0d(1-p)]^2} - \frac{\cos[k_0d(1+p)]}{k_0d(1+p)} + \frac{\cos[k_0d(1-p)]}{k_0d(1-p)}\right\}. \quad (16)$$

The values of ϵ_{1s} for $p = 1$

$$\epsilon_{1s} = \frac{3}{(2k_0d)^2}\left[\frac{\sin 2k_0d}{2k_0d} - \cos 2k_0d\right] \quad (17)$$

are given in Fig. 2. For small values of k_0d, $\epsilon_{1s} \approx 1$ is independent of p.

2. A Group Consisting of Two Complex Spherical Radiators

Let us examine a group consisting of two complex (i.e., nonprimitive) spherical radiators which are positioned in the same manner as the group of primitive radiators (Fig. 1) examined earlier.

The normalized directionality coefficient for an isolated radiator can be written in the form

$$\mathcal{S}_1(\theta) = \frac{\sum_n C_nP_n(\cos\theta)}{\sum_n C_n}. \quad (18)$$

It should be noted that C_n are the coefficients for expanding the angular distribution at a distance far from the system into a series consisting of Legendre polynomials. The angular distribution at the surface of the radiator can be written in the form of a series:

$$V(\theta) = \sum_n e_nP_n(\cos\theta). \quad (19)$$

Here

$$e_n \neq \alpha C_n,$$

where α is a constant. The latter inequality means that the angular distribution at the surface of a complex spherical radiator does not coincide with the angular distribution at a distance far from the radiator [3].

Taking Equation (18) into account, we find the power which is radiated by each radiator, where

$$W_{gr} = \frac{4\pi r^2 I_0}{(\sum_n C_n)^2}[D_1 \pm (D_2 + D_3)], \quad (20)$$

$$\left.\begin{aligned} D_1 &= \sum_n \frac{C_n^2}{2n+1}, \\ D_2 &= \frac{1}{2}\sum_n C_n^2 \int_{-1}^{1} P_n^2(y)\cos 2ay\,dy = \sum_n \Delta_{n,n}, \\ D_3 &= \frac{1}{2}\sum_{n\neq m} C_n C_m \int_{-1}^{1} P_n(y) P_m(y) \cos 2ay\,dy = \sum_{n\neq m} \Delta_{n,m} \end{aligned}\right\}. \tag{21}$$

The plus and minus signs in Equation (20) refer respectively to in-phase and phase-opposition groups.

The power radiated by an isolated complex radiator can be written in the form

$$W = \frac{4\pi r^2 I_0}{\left(\sum_n C_n\right)^2} D_1. \tag{22}$$

Therefore the ratio between the resistive component of the radiation impedance of the complex spherical radiator in the group and the resistive component of the radiation impedance of the radiator when it is detached will be

$$\frac{R_{s\,\mathrm{gr}}}{R_s} = 1 \pm \left(\frac{D_2}{D_1} + \frac{D_3}{D_1}\right). \tag{23}$$

The resulting relationships can be analyzed in the following manner. A complex spherical source, according to Equation (19), is an aggregate of primitive sources of various orders. Therefore the resistance introduced into a given source by another source can be represented in the form of the sum of the resistances of primitive radiators of n different orders which are introduced into the given source by the primitive radiators which make up the other source and which have the same n different orders; the resistance can also be represented in the form of the sum of the resistances of primitive radiators of n different orders, which are introduced into the given source by primitive radiators of other orders $m \neq n$. The mutual resistances of the primitive components of the same order are determined by the quantities $\Delta_{n,n}$, and the mutual resistances of primitive components of different orders are determined by the quantities $\Delta_{n,m}$.

Since $P_n(y)$ is an even function of y when n is even and is an odd function of y when n is odd, it follows that

$$\Delta_{n,m} = \frac{1}{2} C_n C_m \int_{-1}^{1} P_n(y) P_m(y) \cos 2ay\,dy$$

is identically equal to zero when the sum n + m is odd.

Thus when n + m is equal to an odd number,

$$\frac{R_{s\,\mathrm{gr}}}{R_s} = 1 \pm \frac{D_2}{D_1}. \tag{24}$$

Thus for a radiator which is of the zero-plus-one order we obtain

$$\frac{R_{(0+1)\mathrm{gr}}}{R_{(0+1)}} = \frac{C_0^2(1 \pm \varepsilon_0) + \frac{C_1^2}{3}(1 \pm \varepsilon_1)}{C_0^2 + \frac{C_1^2}{3}},$$

where ϵ_0 and ϵ_1 are determined by Expressions (9) and (10).

For a radiator with a cardidoid directional characteristic, $C_1 = C_0$ (but $e_1 \gg e_0$); here

$$\frac{R_{\mathrm{car\,gr}}}{R_{\mathrm{car}}} = 1 \pm \frac{3}{4}\left(\varepsilon_0 + \frac{1}{3}\varepsilon_1\right). \quad (25)$$

As we know, at low frequencies a half-piston can be approximately represented by a spherical radiator of zero-plus-one order under conditions where $e_0 = e_1$ (i.e., provided that $C_0 \gg C_1$ [3]). Consequently for a half-piston it follows that

$$\frac{R_{\mathrm{f.o.\,gr}}}{R_{\mathrm{f.o.}}} \approx 1 \pm \varepsilon_0. \quad (26)$$

in the low-frequency region (i.e., in the region which is of primary interest).

3. Certain Groups of Radiators

Let us represent the resistive component of the radiation impedance of each radiator of a group consisting of two radiators in the form:

$$R_i = R_{i1} + R_{i2},$$

where $i = 1, 2$, R_{ii} is the intrinsic radiation resistance of the i-th radiator, R_{ij} is the resistance which is produced in the i-th radiator by the j-th radiator.

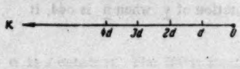


Fig. 3

Taking (9) into account, we obtain the total resistance R_0 of the zero-th radiator of a semi-infinite group (Fig. 3) of in-phase point sources which have an identical given intensity; the expression for this total resistance is

$$R_0 = \sum_{i=0}^{\infty} R_{0i} = R_{00}(1 + A_\infty), \quad (27)$$

where

$$A_\infty = \sum_{i=1}^{\infty} \frac{\sin ikd}{ikd}, \quad (28)$$

d is the distance between adjacent radiators.

The calculation of the sum given by Expression (28) leads to the following values:

$$\left.\begin{aligned} A_\infty &= \frac{\pi - kd}{2kd} \quad \text{for} \quad 0 < kd < 2\pi, \\ A_\infty &= \frac{3\pi - kd}{2kd} \quad \text{for} \quad 2\pi < kd < 4\pi \end{aligned}\right\} \tag{29}$$

etc.

For a finite number (m) of radiators,

$$R_0 = R_{00}(1 + A), \tag{30}$$

where

$$A = \sum_{i=1}^{m-1} \frac{\sin ikd}{ikd}. \tag{31}$$

The values of A for 2, 3, 4, 5, 6 and 7 radiators, as well as the values of A_∞, are given in Fig. 4.

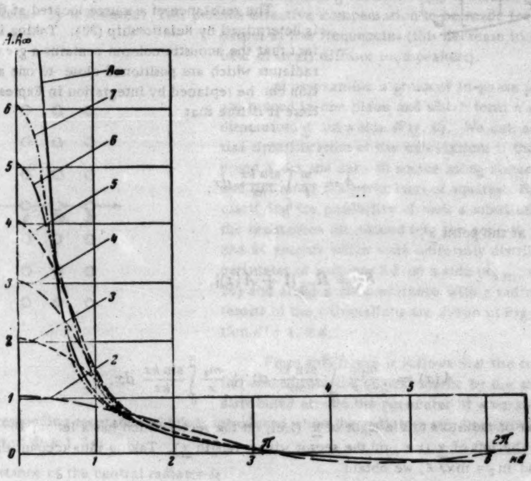


Fig. 4. Values of A_∞, as well as of A, for the values of m indicated by the numbers. For kd = 0, A = m−1; for kd > 0, A < m−1.

It is interesting to note that A and A_∞ differ appreciably from one another only for values of kd which are less than a magnitude of the order of $1/(m-2)$.

For the m -th radiator of a semi-infinite group,

$$R_m = R_{mm}(1 + A + A_\infty).$$

For any radiator in an infinite linear group of in-phase radiators,

$$R_i = R_{ii}(1 + 2A_\infty). \tag{32}$$

From (32) it follows that any source in an infinite group of sources operating in a regime of constant wave velocity will radiate an amount of power which differs by an amount

$$N_1 = 10 \log(1 + 2A_\infty) \text{ db} \tag{33}$$

from the power radiated by an isolated source. The values of N_1 are given in Fig. 5.

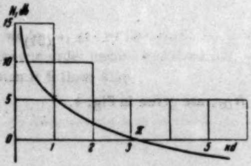


Fig. 5

Let us pass on to an examination of acoustic columns (i.e., a linear group of in-phase radiators which are positioned at small distances d from one another).

Let us define the number of radiators in the column by m and the over-all length of the column by ℓ. In practice the derivations made below for small radiators of the zero-th order can also be used for half-pistons.

The resistance of a source located at the point x = 0 is determined by Relationship (30). Taking into account the fact that the acoustic column contains a great number of radiators which are positioned close to one another, summation can be replaced by integration in Expression (31) for A. Here it is true that

$$A = \frac{m}{l}\int_0^l \frac{\sin kx}{kx}\,dx. \tag{34}$$

For a radiator located at the point x,

$$R_x = R_{xx}[1 + A(x)], \tag{35}$$

where

$$A(x) = \frac{m_1}{l - x}\int_0^{l-x} \frac{\sin kx}{kx}\,dx + \frac{m_2}{x}\int_0^{x} \frac{\sin kx}{kx}\,dx. \tag{36}$$

Here m_1 is the number of radiators to the right of x (i.e., on the sector which has a length $\ell - x$), m_2 is the number of radiators to the left of x (i.e., on the sector with a length x). Taking into account the fact that $m_1 = m(\ell - x)/\ell$ and $m_2 = mx/\ell$, we obtain

$$A(x) = \frac{m}{kl}[\operatorname{Si}(kl - kx) + \operatorname{Si} kx]. \tag{37}$$

The value of R_x averaged over x will be

$$\overline{R_x} = R_{xx}[1 + \overline{A(x)}], \tag{38}$$

where

$$\overline{A(x)} = \frac{m}{kl^2}\int_0^l [\mathrm{Si}\,(kl - kx) + \mathrm{Si}\,kx]\,dx,$$

this reduces to

$$\overline{A(x)} = m\,2\left[\frac{\mathrm{Si}\,kl}{kl} + \frac{\cos(kl-1)}{(kl)^2}\right]. \tag{39}$$

The power radiated by m isolated sources is proportional to mR_{xx}, whereas the same sources in the column radiate a power which is proportional to $m\overline{R}_x$. Thus, when m sources are placed in a column, the power increases by an amount

$$N_2 = 10 \log [1 + \overline{A(x)}]\ \mathrm{db} \tag{40}$$

The values of $\overline{A(x)}/m$ are given in Fig. 2.

It should be noted that acoustic columns have a distinguishing feature: $\overline{A(x)}$ increases practically monotonically when the frequency is lowered. This permits effective compensation to be made for the decrease in output at low frequencies (this decrease in output is characteristic of small diffusor loudspeakers).

Let us examine a group of in-phase point sources which are placed in one plane and which form a square grid with dimensions d on a side (Fig. 6). We can achieve a substantial simplification of the calculations if the sources are disposed about the zero-th source along concentric circumferences and not along the perimeters of squares. For the purpose of clarifying the possibility of such a substitution we calculated the resistances introduced into the zero-th radiator by 8, 16 and 24 sources which were uniformly distributed around the perimeter of a square 2d on a side ($R_{0\,m}$, where m = 8, 16, 24) and along a circumference with a radius d' ($R'_{0\,m}$). The results of the calculations are shown in Fig. 7 for the condition d' = 1, 2d.

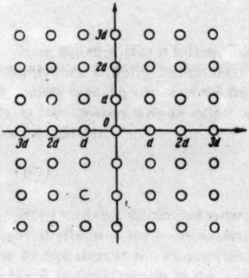


Fig. 6

From this figure it follows that the resistance which is introduced into the central source by the sources uniformly distributed around the perimeter of a square are practically the same as the corresponding resistance which is introduced when the radiators are distributed around a circumference with a radius of d' = 1, 2d.

The total resistance of the central radiator is

$$R_0 = R_{00} + 8R_{0\,d'} + 16R_{02d'} + 24R_{03d'} + \cdots,$$

where $R_{0\,id}$ is the resistance into which is introduced the zero-th source by one source located at a distance id', or

$$R_0 = R_{00}\,(1 + B_\infty), \tag{41}$$

where

$$B_\infty = \frac{8}{kd'} \sum_{i=1}^{\infty} \sin ikd'.$$

An expression for B_∞ which is convenient for making calculations can be obtained when the introduced resistances are determined while taking into consideration the damping of the wave in air. Let us assume that the amplitude of the velocity potential is attenuated according to the law

$$\Phi_m = \Phi_{m0} e^{-\alpha r}, \tag{42}$$

where α is the damping index and $\underline{r}$ is the distance.

Taking (42) into account and proceeding as indicated in [4], we obtain the values of the mutual resistances for two identical point sources:

$$R_{12} = R_{21} - R_{11}\varepsilon_0 e^{-\alpha d} = R_{22}\varepsilon_0 e^{-\alpha d}, \tag{43}$$

whence it follows that

$$B_\infty = \frac{8}{kd'} \sum_{i=1}^{\infty} e^{-i\alpha d'} \sin ikd'$$

or

$$B_\infty = \frac{4 \sin kd'}{kd'} \frac{1}{\operatorname{ch} \alpha d' - \cos kd'}. \tag{44}$$

For small values of kd' and $\alpha d'$,

$$B_\infty = \frac{8}{k^2 d'^2 + \alpha^2 d'^2}. \tag{45}$$

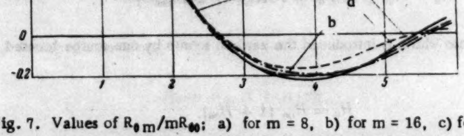


Fig. 7. Values of R_{0m}/mR_{00}; a) for m = 8, b) for m = 16, c) for m = 24, as well as the values of R^*_{0m}/mR_{00} (curve $\underline{d}$); for curve $\underline{d}$, values of $\underline{kd'}$ are plotted along the axis of abscissas, and for curves $\underline{a}$, $\underline{b}$, and $\underline{c}$ the values of $\underline{kd}$ are plotted along the axis of abscissas also.

Each radiator of the group which forms an unlimited square grid can be viewed as a central radiator; therefore the power radiated by each source of the group differs from the power radiated by an isolated source by an amount which is approximately equal to

$$N_3 = 10 \log (1 + B_{\infty}) \text{ db} \tag{46}$$

The values of N_3 for $\alpha d' \rightarrow 0$ are shown in Fig. 8.

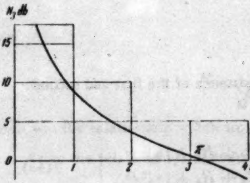


Fig. 8

When damping is taken into account we have the following expression instead of the Equation (29) for linear groups of point radiators (when $0 < kd < 2\pi$):

$$A_{\infty} = \frac{1}{kd} \operatorname{arc} \tan \frac{\sin kd}{e^{\alpha d} - \cos kd} .$$

For small kd and $\alpha d \ll (kd)^2/2$,

$$A_{\infty} \rightarrow \frac{\pi}{2kd} . \tag{47}$$

However, if kd is small so that $\alpha d \gg kd$, then

$$A_{\infty} \rightarrow \frac{1}{\alpha d} .$$

From Equation (45) it follows that for a surface group of radiators the compensation of frequency distortion is possible when utilizing loudspeakers that have an output at the low frequencies which does not attenuate any more rapidly than the square of the frequency, whereas for a linear group compensation is only possible when utilizing loudspeakers with an output which does not attenuate more rapidly than the first power of frequency at low frequencies.

SUMMARY

When utilizing distributed systems of radiators, acoustic columns, etc., it is necessary to take into account the mutual effects of the loudspeakers; these effects lead to a considerable variation in the radiation resistance of the loudspeakers at low frequencies. When the radiators are placed close enough to one another, it is possible to achieve an improvement in the output at low frequencies in comparison to the output which results at these frequencies when isolated loudspeakers are used.

This fact explains why it is possible to obtain a satisfactory output from a distributed system at low frequencies when small diffusor loudspeakers (whose isolated characteristics are distinguished by poor low-frequency output *) are used. The distance between radiators should be selected on the basis of the necessary amount of correction at low frequencies.

However it should be kept in mind that a decrease in the output is possible at certain frequencies, and therefore the calculation must be made not only for the lowest frequency region but for other regions as well.

A very favorable property of acoustic columns should be noted here — the fact that $\overrightarrow{A(x)}$ increases practically monotonically when the frequency is lowered.

---

* Certain practical examples of this phenomenon were cited to the author by B. G. Belkin.

APPENDIX

In Section 2 we showed that when n + m was equal to an odd number, the mutual resistances are determined solely by the interactions between components of the same order in different sources. For a radiator of the zero-plus-one order this situation can be additionally explained in the following manner.

Let us examine two radiators I and II of the zero-plus-one order (Fig. 9). Each of these radiators can be represented in the form of two phase-opposition point sources 1, 2 or 3, 4 which form dipoles, and by a point source 5 or 6. The total resistance of each point source is

$$R_i = \sum_{j=1}^{6} R_{ij},$$

where $i = 1, \ldots, 6$.

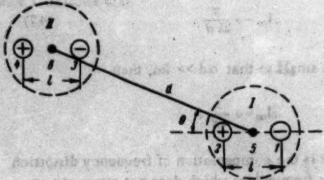


Fig. 9

The total resistance of the first and second complex radiators is

$$\left.\begin{aligned} R_{\rm I} &= R_1 + R_2 + |\gamma|^2 R_5, \\ R_{\rm II} &= R_3 + R_4 + |\gamma|^2 R_6, \end{aligned}\right\} \qquad (48)$$

where γ is a coefficient which determines the ratio between the intensities of the point source (5 or 6) and the sources which form the dipoles.

The total resistance of a system consisting of two complex radiators (I and II), referred to one of the radiators, is

$$R_{(0+1)\,{\rm gr}} = \frac{R_{\rm I} + R_{\rm II}}{2}. \qquad (49)$$

In calculating $R_{(0+1)\,{\rm gr}}$ it should be taken into account that

$$\left.\begin{aligned} R_{d{\rm I}} &= \sum_{j=1}^{4} (R_{1j} + R_{2j}), \\ R_{d{\rm II}} &= \sum_{j=1}^{4} (R_{3j} + R_{4j}), \\ R_{\tau{\rm I}} &= R_{55} + R_{56}, \\ R_{\tau{\rm II}} &= R_{65} + R_{66}, \end{aligned}\right| \qquad (50)$$

where $R_{d{\rm I}}$ is the total resistance of dipole I (which consists of radiators 1 and 2) including the reaction of dipole II (which consists of the radiators 3 and 4) upon it; $R_{d{\rm II}}$ is the same quantity with respect to dipole II including the reaction of dipole I upon it; $R_{T{\rm I}}$ is the total resistance of the source 5 including the reaction of radiator 6 upon it; $R_{T{\rm II}}$ is the same quantity with respect to the source 6 including the reaction of source 5 upon it; it is also true that

$$R_{d{\rm I}} = R_{d{\rm II}}, \quad R_{\tau{\rm I}} = R_{\tau{\rm II}}.$$

Since sources 1 and 2, as well as 3 and 4 operate in phase-opposition, it follows that

$$R_{15}+R_{25}=0, \quad R_{51}+R_{52}=0,$$
$$R_{36}+R_{46}=0, \quad R_{63}+R_{64}=0.$$

Further, taking into account that sources 1 and 4, as well as 2 and 3, operate in phase-opposition, we obtain

$$R_{16}+R_{45}=0, \quad R_{61}+R_{54}=0,$$
$$R_{26}+R_{35}=0, \quad R_{62}+R_{65}=0.$$

Thus

$$R_{(0+1)_{\mathrm{gr}}}=R_{\mathrm{dI}}+R_{\tau\mathrm{I}}\,|\gamma|^2=R_{\mathrm{dII}}+R_{\tau\mathrm{II}}\,|\gamma|^2, \tag{51}$$

which was the relationship which we were required to prove.

From the relationships derived here it also follows that in the general case

$$R_{\mathrm{I}} \neq R_{\mathrm{II}} \neq R_{(0+1)_{\mathrm{gr}}}$$

When there is interaction between two point sources of identical intensity it is also not always true that $R_1 = R_2$. This problem was examined in another paper [4] where the mutual resistances of point sources were examined for any phase relationships.

Making use of Expression (50) it can be proven that when $d \gg \ell$ and $\sqrt{d\lambda/\ell} \gg 1/2$ (Fig. 9),

$$R_{\mathrm{dI}}=R_{\mathrm{d}}\,(1+\varepsilon_1\cos^2\theta), \tag{52}$$

where R_d is the isolated radiation resistance of dipole I. When $\theta = 0$ (or π), Equation (52) is real for any d/λ.

LITERATURE CITED

[1] A. A. Kharkevich, "Equivalent circuits and characteristics for spherical antennas," J. Tech. Phys. (USSR) 11, 8, 746-752 (1941).

[2] M. I. Karnovskii, "The problem of complex signal interference," Trans. Acoust. Commission Acad. Sci. USSR 8, 139-150 (1955).

[3] M. I. Karnovskii, "Directional properties of spherical radiators," J. Tech. Phys. (USSR) 10, 12, 970-979 (1940).

[4] M. I. Karnovskii, "Mutual acoustic impedances of spherical radiators and resonators," Doklady Akad. Nauk SSSR 32, 1, pp. 40-43 (1941); The Calculation of Mutual Acoustic Impedances, Vol. I (Collected Works of the Kiev Cinema-Engineering Institute, 1948), pp. 95-113.

Kiev Polytechnical Institute

Received October 11, 1955.

CORRELATION METHOD FOR MEASURING THE DISTORTION COEFFICIENT OF TRANSMISSION

M. A. Sapozhkov

A method is proposed for measuring the generalized distortion and interference coefficient, which is based upon utilizing the mutual correlation between the input and output processes.

The existing methods for measuring distortion and interference only indirectly characterize those types of distortion and interference which occur under actual operating conditions in channels which are used for communication, broadcasting, television, etc. Besides, the quantities which are obtained as a result of such measurements (the "clear factor", the nonuniformity of the frequency response, the signal-to-interference ratio etc.) cannot be combined into one whole which characterizes the various distortion factors of transmission and takes into account all of the distortion and interference. Finally, a serious shortcoming of these methods is the fact that these measurements cannot be made during the actual operation of the channel.

The task of this present paper is to propose a new method for measuring these quantities, which makes it possible to determine coefficients which more accurately and completely characterize the interference and distortion which actually occur in communications channels under operating conditions. The correlation method can serve as the answer to these problems; this method is based upon utilizing the mutual correlation between the input and output processes.

In accordance with the method for determining the correlation coefficient,

$$Q=\frac{\overline{(x-\bar{x})\,(y-\bar{y})}}{\sqrt{\overline{(x-\bar{x})^2}\;\overline{(y-\bar{y})^2}}}=\frac{\overline{xy}-\bar{x}\bar{y}}{\sigma_x\sigma_y}, \qquad (1)$$

where $\underline{x}$ and $\underline{y}$ are two processes between which some arbitrary relationship may exist. In the given case one of them (for example, $\underline{x}$) may be the input process and the second quantity $\underline{y}$ may be the output quantity.

In accordance with this reasoning $\underline{x}$ and $\underline{y}$ are the mean (statistically speaking) values of $\underline{x}$ and $\underline{y}$, σ_x and σ_y are their standards, and $\overline{xy}$ is the correlation function. When $\bar{x}=\bar{y}=0$,

$$Q=\frac{\overline{xy}}{\sigma_x\sigma_y}. \qquad (2)$$

From Equations (1) and (2) it follows that the correlation coefficient Q expresses a ratio between the effective value of those output components of the device which correspond to input components, and the overall effective value which appears at the output. Here it should be noted that one or another time displacement may exist between the input process and the output process which corresponds to it; this is due to the propagation constant, and therefore in Expressions (1) and (2) $\underline{y}$ must be assumed to represent the output process when no such time displacement is present.

Let the following perturbation appear at the input of the device:

$$x=\sum_{i=1}^{n} x_i, \tag{3}$$

where x_1, x_2, etc. are components which are not correlated among one another. Then we shall have the following expression at the output for the case of linear systems:

$$y = y_{\text{in}} + y_{\text{int}} + \sum_{i=1}^{n} K_i x_i, \tag{4}$$

where y_{int} is the interference which is introduced into the device beyond its input, y_{in} is the intrinsic process of the device, $K_i = \alpha_i + j\beta_i = \gamma_i e^{j\varphi_i}$ is the sensitivity coefficient of the device with respect to the i-th process (after the propagation constant has been deducted).

The correlation coefficient which corresponds to this situation is

$$Q=\frac{\sum\limits_{i=1}^{n} \alpha_i \overline{x_i^2}}{\sqrt{\sum\limits_{i=1}^{n} \overline{x_i^2}\left(\overline{y_n^2}+\overline{y_u^2}+\sum\limits_{i=1}^{n} \gamma_i^2 \overline{x_i^2}\right)}}. \tag{5}$$

Let us examine particular cases. When neither the transient nor the steady-state processes produce any distortion, but when interference is present, the correlation coefficient is

$$Q=\frac{K\sum\limits_{i=1}^{n} \overline{x_i^2}}{\sqrt{\sum\limits_{i=1}^{n} \overline{x_i^2}\left(y_u^2+K^2\sum\limits_{i=1}^{n} \overline{x_i^2}\right)}}=\frac{1}{\sqrt{1+\nu_{\text{int}}^2}}, \tag{6}$$

where K is the sensitivity coefficient of the channel, ν_{int} is the interference coefficient

$$\nu_{\text{int}}=\frac{\sigma_{\text{int}}}{\sigma_s}=\frac{1}{K}\sqrt{\frac{\overline{y_{\text{int}}^2}}{\sum\limits_{i=1}^{n} \overline{x_i^2}}}, \tag{7}$$

where σ_{int} is the mean square value of the interference at the output of the device and σ_s is the mean square value of the signal at the output (i.e., the correlation coefficient is single-valuedly related to the interference coefficient).

If the device includes a nonlinearity and there is no interference or linear distortion, then the correlation coefficient is always less than one because modulation components are formed. The latter components can be viewed as interference, and as a consequence it follows that

$$Q = \frac{1}{\sqrt{1+\nu_{\mathrm{d}}^2}}, \tag{8}$$

where ν_{d} is the generalized coefficient of nonlinear distortion

$$\nu_{\mathrm{d}} = \frac{\sigma_{\mathrm{d}}}{\sigma_s} = \frac{1}{K}\sqrt{\frac{\overline{y_{\mathrm{d}}^2}}{\sum\limits_{i=1}^{n}\overline{x_i^2}}}, \tag{9}$$

y_{d} is the process which is formed by the joint effect of all of the modulation components, σ_{d} is the mean-square value of all of the modulation components.

In the particular case when there is a sinusoidal input perturbation the coefficient ν_n is an ordinary "clear factor":

$$\nu_{\mathrm{d}} = \frac{\sqrt{y_2^2 + y_3^2 + \dots}}{y_1}, \tag{10}$$

and for fluctuating noise with a normal distribution for a dynamic characteristic

$$y = ax^{2k-1}$$

we have [on the basis of Equation (2)] the following expression for Q:

$$Q = \frac{\overline{x^{2k}}}{\sqrt{\overline{x^2}\cdot\overline{x^{4k-2}}}} = \frac{\int\limits_{-\infty}^{\infty} x^{2k} e^{-\frac{x^2}{2}}\,dx}{\sqrt{\int\limits_{-\infty}^{\infty} x^2 l^{-\frac{x^2}{2}}\,dx \cdot \int\limits_{-\infty}^{\infty} x^{4k-2} e^{-\frac{x^2}{2}}\,dx}} = \frac{(2k-1)!!}{\sqrt{(4k-3)!!}}, \tag{11}$$

whence the coefficient of nonlinear distortion is

$$\nu_{\mathrm{d}} = \frac{\sqrt{(4k-3)!! - [(2k-1)!!]^2}}{(2k-1)!!}. \tag{12}$$

This is a generalized coefficient of nonlinear distortion, which is determined for a discrete spectrum with a large number of components that have identical amplitudes but frequencies which are irrationally related to one another [1].

Therefore in the case of nonlinear distortion it is also true that the coefficient of mutual correlation is related in a single-valued manner to the distortion coefficient. The coefficient of transient distortion can also be determined analogously and is represented by the following expression:

$$\nu_{in} = \frac{\sigma_{in}}{\sigma_s} = \frac{1}{K}\sqrt{\frac{\overline{y_{in}^2}}{\sum\limits_{i=1}^{n}\overline{x_i^2}}}, \tag{13}$$

where σ_{in} is the mean-square value of the intrinsic process of the device.

When interference, transient processes and nonlinear distortion operate jointly under the conditions for which the effects of frequency and phase distortion * can be neglected, the coefficient of mutual correlation determines the generalized coefficient of interference and distortion; this coefficient is the ratio of the mean-square value of all of the interference and of the distortion products to the mean-square value of the signal at the output:

$$\nu = \frac{\sigma}{\sigma_s} = \frac{1}{K}\sqrt{\frac{\overline{y_{int}^2}+\overline{y_{in}^2}+\overline{y_d^2}}{\sum\limits_{i=1}^{n}\overline{x_i^2}}}. \tag{14}$$

Let us examine the case when the effect of interference, nonlinear distortion and transient distortion can all be neglected. For the purpose of greater simplicity let us take a sinusoidal component in the capacity of x_i; then the correlation coefficient will be

$$Q = \frac{\sum\limits_{i=1}\alpha_i\overline{x_i^2}}{\sqrt{\sum\limits_{i=1}^{n}\overline{x_i^2}\left(\sum\limits_{i=1}^{n}\alpha_i^2\overline{x_i^2}+\sum\limits_{i=1}^{n}\beta_i^2\overline{x_i^2}\right)}} = \frac{\sum\limits_{i=1}^{n}\gamma_i\overline{x_i^2}\cos\varphi_i}{\sqrt{\sum\limits_{i=1}^{n}\overline{x_i^2}\cdot\sum\limits_{i=1}^{n}\gamma_i^2\overline{x_i^2}}}. \tag{15}$$

And analysis of this expression shows that included in the correlation function are only those components of the output process which are in phase with the input components. Therefore when phase shifts $\varphi_i \neq 0$ * are present, the components $\sum\limits_{i=1}^{n}\gamma_i^2\overline{x_i^2}\sin^2\varphi_i$, which are in quadrature with the corresponding input components play the part of a certain kind of interference which is produced by phase distortion; this is significant, for example, in television transmission.

---

* As will be shown below, their effect upon the correlation is small.

** Since phase distortion occurs only when the frequency dependence of the phase shifts deviates from a straight line relationship, it follows that the quantity φ_i should be assumed to represent solely the difference between the actual phase shift and the phase shift which is proportional to the frequency. The latter phase shift includes the phase shift which is produced by the time shift caused by the propagation constant of the waves.

In broadcasting and communications, phase distortions do not play a role because the ear cannot detect them; because of this it would be better to determine the correlation coefficient for these types of transmission without taking phase shifts into account (i.e., the correlation function should be determined as

$$\sum_{i=1}^{n} \gamma_i \overline{x_i^2} \, . \tag{16}$$

When there is no appreciable phase distortion Equation (15) is of the form:

$$Q_i = \frac{\sum\limits_{i=1}^{n} \gamma_i \overline{x_i^2}}{\sqrt{\sum\limits_{i=1}^{n} \overline{x_i^2} \sum\limits_{i=1}^{n} \gamma_i^2 \overline{x_i^2}}} \, . \tag{17}$$

In the particular case when there is one component, Q = 1. If all of the components have identical effective values, then

$$Q = \frac{\overline{\gamma_i}}{\sqrt{\overline{\gamma_i^2}}} \leqslant 1. \tag{18}$$

When the components are not identical in magnitude the correlation coefficient will always satisfy the condition

$$1 > Q \geqslant \frac{\overline{\gamma_i}}{\sqrt{\overline{\gamma_i^2}}} \tag{19}$$

and will be the ratio of the weighted magnitude of the average sensitivity coefficient to the mean-square value of this coefficient (i.e.,

$$Q = \frac{\overline{\gamma_{wt}}}{\sqrt{\overline{\gamma_{wt}^2}}} \tag{20}$$

therefore the correlation coefficient indirectly characterizes the frequency distortion. This postulate can also be illustrated in the following manner. Replacing the sensitivity coefficient γ_i in Expression (17) by the difference $\gamma_{max} - \Delta\gamma_i$ (where γ_{max} is the maximum value of the gain coefficient), we obtain:

$$Q = \frac{\gamma_{max} \sum\limits_{i=1}^{n} \overline{x_i^2} - \sum\limits_{i=1}^{n} \Delta\gamma_i \overline{x_i^2}}{\sqrt{\sum\limits_{i=1}^{n} \overline{x_i^2} \sum\limits_{i=1}^{n} (\gamma_{max} - \Delta\gamma_i) \overline{x_i^2}}} \, . \tag{21}$$

From the above expression it follows that when frequency distortion is present it is as if interference which is correlated with the signal appeared; as a consequence of this the correlation function and the effective value of the output effect both decrease correspondingly, whereas it would be more proper to characterize frequency distortion solely by a decrease in the correlation function.

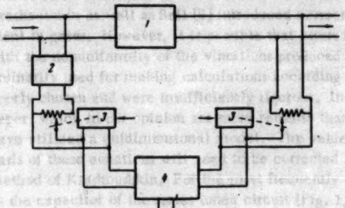
The correlation coefficient can easily be determined by means of measuring the mutual correlation function of the input and output processes while maintaining the mean-square values of these processes constant. In doing so such measurements may be conducted under normal operating conditions for the device (it is even better to do so), since the measurements then yield practical values for the coefficients.

Since the mean (statistically speaking) value and the time average (or the value which is averaged over the nonstationary processes) are only equal to one another under conditions where the averaging time is infinite (i.e., where

$$\overline{z} = \lim_{T\to\infty}\int_0^T z\,dt \text{ and } \overline{z^2} = \sigma_z^2 = \lim_{T\to\infty}\int_0^T z^2\,dt$$

it follows theoretically that the measurement of the mutual correlation coefficient must be made with an instrument that has an infinitely large time constant. It is evident that the magnitudes of the time constant must in such a case be selected in the same manner as they are for any other measurements of the average values of noise, interference, speech, music, etc. (i.e., they should be chosen as large as possible). The practical limit is reached for a time constant of about 15 sec. A time constant greater than this causes inconvenience in measurement. It should merely be kept in mind that the values of a coefficient measured under such conditions vary continuously and that therefore they must either be averaged or the limits over which they vary must be determined.

In order to measure the mutual correlation coefficient it is possible to utilize a device consisting of a multiplier, two mean-square value stabilizers and a phase compensator (see figure). In this circuit the instrument at the output of the multiplier will produce readings which are proportional to the correlation coefficient. However, two features of this circuit should be discussed. In the first place, the time constant of the mean square stabilizers and of the multiplier together with the meter must be sufficiently great; in the second place, the effect of phase shift (and in the general case the effects of the presence of time delay) between the the output and input processes must be eliminated. The time constant of the mean-square value stabilizers can be made practically any desired value, if, for example, devices consisting of a squaring element which is loaded with a RC circuit are used in the capacity of stabilizers. Analogously, the time constant of the multiplier may be made practically any desired value if the multiplier consists of a mixer (of the type used in superheterodyne receivers) loaded by a RC circuit. The large time constant can also be achieved by using an appropriate meter.



Circuit for the correlation unit: 1) channel, 2) phase compensator, 3) automatic stabilizers, 4) multiplier, Q) correlation-coefficient meter.

The specified device reacts to phase shifts between x and y, and therefore a phase compensator is utilized in order to eliminate phase shifts which are proportional to the frequency; this compensator has a sensitivity characteristic which is independent of frequency. In all cases the compensation of phase shifts must be carried out before the maximum values of the correlation coefficient are obtained. In the general case, when lag exists between the output process and the input process, a delay line must be connected in place of the phase compensator. Since phase distortion will occur under these conditions as well, the resulting correlation coefficient is somewhat lowered with respect to the requirements of broadcasting and communications. In order to eliminate the effects of phase distortion the mixing device should not react to phase shifts between the waves which are applied to it.

In conclusion let us indicate that the method for measuring the mutual correlation coefficient also has the advantage that it yields a generalized interference and distortion coefficient and permits the measurement of the values of these coefficients under normal operating conditions for the devices and for any type of transmission. The shortcomings of the method include an incomplete consideration of the effects of frequency distortion; this results in correlation coefficient values which are somewhat lowered with respect to those required by broadcasting and communications systems. The latter shortcoming can be eliminated by utilizing more complex measuring circuits.

As far as the sensitivity of this method is concerned, it follows that according to the concepts discussed above the sensitivity is the same as, for example, that which obtains in measuring the coefficient of nonlinear distortion.

LITERATURE CITED

[1] M. A. Sapozhkov, Doctoral Dissertation, (VKIAS, 1954).

Moscow Received January 5, 1955.

THE THEORY OF AN ULTRASONIC INTERFEROMETER

V. A. Solov'ev

New design formulas are proposed for calculating the absorption coefficient of ultrasonic waves from interferometer measurements.

The ultrasonic interferometer is one of the most widely utilized instruments for measuring the velocity of sound in gases and liquids. The accuracy of the interferometer method, notwithstanding its comparative simplicity, is very high: only certain special variants of the pulse method which make use of extremely complex radio-engineering circuits can compete with the interferometer in terms of the accuracy of measuring sound velocity. In those cases where the absorption is not too small the interferometer can be utilized for the measurement of sound absorption as well. Here also, an important advantage of the interferometer method is the simplicity of the unit which can be used and, most important, the possibility of measuring the velocity and absorption of sound simultaneously. However in practice the interferometer has been used for absorption measurements almost exclusively in the case of gases. The problem consists of the fact that such measurements have often proved to be in error and that in the case of liquids it is especially difficult to eliminate these errors. Reliable measurements of absorption by means of an interferometer require a highly developed theoretical analysis of the instrument. Such a theory has been developed by many authors, but it still requires a number of refinements.

In measuring the absorption of sound by means of an interferometer a very serious difficulty exists in principle: a piezocrystalline plate usually executes vibrations which are not piston-like but are extremely complex and nonuniform. This can greatly distort the results of absorption measurements and even the results of sound velocity measurements. Krashnushkin [1] developed a method which permits the detection and elimination of errors which are associated with the nonuniformity of the vibrations of the plate. By means of this method Krashnushkin as well as Bell [2] introduced corrections into a number of measurements of the absorption coefficient in gases. However, it is possible that errors in measurements of sound absorption were associated not only with the nonuniformity of the vibrations produced by the piezocrystal, but also with the fact that the methods ordinarily used for making calculations according to the formulas of a unidimensional approximation were poorly chosen and were insufficiently rigorous. In this regard several computing formulas are derived in this paper, which in our opinion are more reliable than previously suggested formulas. In deriving the formulas we have utilized a unidimensional model. The values of the absorption coefficient which were calculated on the basis of these equations still need to be corrected for the nonuniformity of the acoustic field according to the method of Krashnushkin. For the most frequently used method of connecting the interferometer circuit – parallel to the capacitor of the series tuned circuit (Fig. 1, a) – the equation for the current in the coil of the tuned circuit is of the form:

$$I = \frac{V_0}{r_0 + 1/\omega^2 C_0^2 \left(r + \frac{\mathfrak{Z} S}{4\beta^2} \frac{1 + \delta e^{-2\alpha l} e^{-4i\pi l/\lambda}}{1 - \delta e^{-2\alpha l} e^{-4i\pi l/\lambda}} \right)}. \tag{1}$$

(the equivalent circuit is shown in Fig. 1, b). Here V_0 is the emf which is excited in the tuned circuit, r_0 is the resistance of the tuned circuit, C_0 is the capacitance including the intrinsic capacitance of the piezocrystal), ω is the operating frequency (see below). The expression in parentheses represents the equivalent electrical impedance of the piezocrystal loaded with a column of liquid (z in the equivalent circuit). The resistance r takes into account all types of resistive losses in the piezocrystal, the electrodes, holders, etc. Further, $\mathcal{Z}$ is the specific acoustic impedance of the investigated liquid, α is the absorption coefficient, λ is the wavelength of the sound, δ is the coefficient of reflection at the boundary between the liquid and the reflector, ℓ is the height of the column of the investigated liquid, S is the radiating area of the crystal, β is the electromechanical transducing coefficient. For the piezocrystalline plate, $\beta = eS/\ell_1$, where e is the piezoelectrical constant, ℓ_1 is the thickness of the plate.

Formula (1) is equivalent to the Hubbard Equations [3]. It should be remembered that this formula is accurate only if the operating frequency ω is chosen according to the method described by Hubbard. This frequency coincides with the frequency of the tuned circuit and is close to the intrinsic frequency of the piezocrystal but is not equal to it. Frequently an oscillator is used whose frequency is governed by a second piezocrystal which is identical to the basic piezocrystal. Then, Formula (1) will only be a good approximation when the condition $1/\omega C_0 \ll z$ obtains.

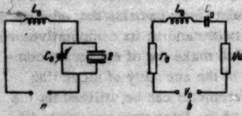


Fig. 1

In order to simplify further calculations let us introduce certain approximate assumptions which have usually been made in implicit form. First of all, we shall assume that the reflection coefficient δ is real. In actual fact δ may be complex, both because of the absorption in the investigated liquid and in the material of the reflector and especially because of the reflection of sound from the rear surface of the reflector. However, taking the complex nature of δ into account would only have led to an insignificant (practically speaking) displacement of the current curve over a distance equal to a certain fraction of a wavelength. Further, we shall assume that the absorption, normalized to a wavelength, is not too great ($\alpha\lambda \ll 1$). This condition can be considered fulfilled in all cases which are of practical interest, since if it is not fulfilled the interferometer measurements are completely impossible. Finally, if $\alpha\lambda \ll 1$, then the reactive portion is small and may be assumed equal to zero.*

Taking these approximations into account, it is easy to prove that the extreme values of the current in the tuned circuit will be real. In order to calculate the absorption coefficient α, it is easiest to utilize precisely these values.

When $4\ell/\lambda = N$, where N is an even number, standing waves with a node at the surface of the crystal are established in the medium, and the current in the coil attains its maximum value:

$$I_{\max} = \frac{V_0}{r_0 + 1/\omega^2C_0^2\left(r + \frac{\mathcal{Z}S}{4\beta^2}\,\frac{1+\delta e^{-2\alpha l}}{1-\delta e^{-2\alpha l}}\right)}. \tag{2}$$

When N is odd, standing waves are established with an antinode at the surface of the crystal; then the current is a minimum:

---

* At the crystal-liquid and liquid-reflector boundaries thermal waves arise which lead to a violation of the "classical" boundary conditions [8, 9]. These thermal waves can be taken into account by varying the parameters δ and $\mathcal{Z}$. However, if in doing so $\mathcal{Z}$ proves to be substantially complex, then all of the existing methods of calculating α will be inapplicable.

$$I_{\min} = \frac{V_0}{r_0 + 1/\omega^2 C_0^2 \left(r + \frac{3S}{4\beta^2}\,\frac{1 - \delta e^{-2\alpha l}}{1 + \delta e^{-2\alpha l}}\right)}. \tag{3}$$

Based on the fact that these quantities depend upon l it is possible to find the absorption coefficient α. However, it is still necessary to eliminate the unknown constants of the instrument. For this purpose it is possible to extrapolate the curve for the maxima and the minima of the current to $l = 0$;

$$I_M = \lim_{l \to 0} I_{\max} = \frac{V_0}{r_0 + 1/\omega^2 C_0^2 \left(r + \frac{3S}{4\beta^2}\,\frac{1+\delta}{1-\delta}\right)}, \tag{4}$$

$$I_m = \lim_{l \to 0} I_{\min} = \frac{V_0}{r_0 + 1/\omega^2 C_0^2 \left(r + \frac{3S}{4\beta^2}\,\frac{1-\delta}{1+\delta}\right)}. \tag{5}$$

If the experimental set-up permits (this is not possible in all cases), it is also possible to measure the magnitude of the current for the case of a remote reflector ($l = \infty$):

$$I_\infty = \lim_{l \to \infty} I_{\max} = \lim_{l \to \infty} I_{\min} = \frac{V_0}{r_0 + 1/\omega^2 C_0^2 \left(r + \frac{3S}{4\beta^2}\right)}. \tag{6}$$

In addition, it is possible to measure the magnitude of the current when there is no liquid in the interferometer:

$$I_0 = \frac{V_0}{r_0 + 1/\omega^2 C^2 r} \tag{7}$$

(in the case of gases it is possible to assume $I_0 = I_m$ in practice).

Combining these quantities, it is possible to eliminate V_0, r_0, r, $\frac{3S}{4\beta^2}$, ωC_0 and δ, and thus to determine the absorption coefficient α. Thus, from Formulas (2)-(5) it follows that

$$\tanh^2 \alpha l = \frac{1\Big/\left(\frac{1}{I_{\min}} - \frac{1}{I_M}\right) - 1\Big/\left(\frac{1}{I_m} - \frac{1}{I_{\max}}\right)}{1\Big/\left(\frac{1}{I_{\max}} - \frac{1}{I_M}\right) - 1\Big/\left(\frac{1}{I_m} - \frac{1}{I_{\min}}\right)} = \frac{(I_M - I_{\max})(I_{\min} - I_m)}{(I_M - I_{\min})(I_{\max} - I_m)}, \tag{8}$$

(here $I_{\max}$ and $I_{\min}$ are found by interpolation for the same value of l).

From Formulas (2), (3), (6) and (7) it is possible to obtain another equation:

$$F(l) = \frac{1 + \delta e^{-2\alpha l}}{1 - \delta e^{-2\alpha l}} = \frac{(1/I_{\min} - 1/I_\infty)(1/I_0 - 1/I_{\max})}{(1/I_0 - 1/I_{\min})(1/I_\infty - 1/I_{\max})} = \frac{(I_\infty - I_{\min})(I_{\max} - I_0)}{(I_{\min} - I_0)(I_{\max} - I_\infty)} \tag{9}$$

and $[F(\ell)-1]/[F(\ell)+1] = \delta e^{-2\alpha\ell}$, so that α is calculated as one-half the tangent of the angle of inclination of the straight line $\ln \frac{F-1}{F+1}$ as a function of ℓ.*

Neither of the proposed formulas apply well for gases, since in view of the small magnitude of δ, $I_{\min} \approx I_m \approx I_\infty \approx I_0$. In this case it is possible to derive the function

$$\Phi(l) = e^{\alpha l} \sinh \alpha l = \text{const} \cdot \frac{1/I_{\max} - 1/I_M}{1/I_\infty - 1/I_{\max}} = \text{const} \cdot \frac{I_M - I_{\max}}{I_{\max} - I_\infty} \tag{10}$$

and find α from the ratio of the values of Φ and for two fixed distances ℓ' and ℓ''. First it is necessary to compose a table for the expression $e^{\alpha(\ell'-\ell'')} \cdot \frac{\sinh \alpha\ell'}{\sinh \alpha\ell''}$ as a function of α. Naturally, the Formula (10) is also applicable to liquids.

Probably it is possible to find still other convenient methods for calculating α. Let us note that in actual fact it is not necessary to operate with small values of ℓ in order to apply these formulas (when small values of ℓ are used distortion associated with the nonuniformity of the sound field plays a large part), or to perform the extrapolation to $\ell = 0$; on the contrary, it is possible to assume any reflector position as the initial one (here only the value of δ is changed in the equations).

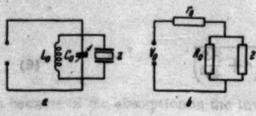


Fig. 2

Usually another method of calculating α is used; this method was proposed by Hubbard [3] for gases and by Fox [5] for liquids. In this method values of the current I in the immediate vicinity of some maximum (for gases) or minimum (for liquids) are used in order to eliminate δ. ** This may cause large errors, since the shape of the current curve is often greatly distorted due to the nonuniformity of the acoustic field [1] and due to the fact that the radiator and reflector are not rigorously parallel [9]. These distortions will have less effect upon the magnitude of the maximums, and therefore the methods of calculation proposed above seem to us to be more reliable. In case it is necessary the value of α which is obtained by one of these methods can be corrected while taking into account the nonuniformities in the vibration of the plate according to the method of Krashnushkin [1].

Mason [6] gives another circuit for connecting the interferometer (Fig. 2, a), which up to now has evidently only been used for measuring the velocity of sounds in liquids [7]. Such a circuit corresponds, for example, to the direct connection of the piezocrystal in parallel with the plate tank of the output tube of the oscillator, which is tuned to its natural frequency. The current through the resistance r_0 is measured (the internal resistance of the tube). The equivalent circuit is shown in Fig. 2, b. Here the current maximums are observed for odd values of $N = 4\ell/\lambda$, and the current minimums are observed for even values of $N = 4\ell/\lambda$. The calculation of the absorption may be made according to any of Formulas (8 - 10) by replacing $I_{\max}$, I_M by

---

* The Beliavskaia Formula [4], which is a particular case of our Formula (9), is valid when

$$r_0 \ll 1/\omega^2 C_0^2 \frac{3S}{4\beta^2} \cdot \frac{1+\delta e^{-2\alpha l}}{1-\delta e^{-2\alpha l}},$$

[i.e., when the resistance of the tuned circuit can be neglected in comparison to the series resistance (the introduced resistance) of the liquid column, even at the current maximum]. In practice this condition is difficult to fulfill.

** These calculations are rather complex, and the details can be obtained from a reading of the papers cited.

$I_{\min}$, I_m and vice versa; when this is done in Formula (10) the value of the constant factor in $\Phi(\ell)$ is changed.* The circuit of Fig. 2 is more convenient, since the resonant impedance R_0 of the parallel tuned circuit $L_0 C_0$ may be made quite large (~ 10^5 ohm) and the current variations obtained are considerably greater. It is only necessary to make use of a power triode with a small internal resistance r_0 in the capacity of an output tube. With an increase in ℓ the peak values of current in the circuit of Fig. 2 vary more rapidly, and thus this circuit permits the measurement of α in liquids where the absorption is less intense.

If the measurements are made by connecting the interferometer into an arm of a high-frequency bridge circuit [6], α may be calculated according to Formula (8) by replacing the current I by the corresponding values of the equivalent susceptance $1/z$ of the piezocrystal. Then it is possible to use the simpler formula

$$F(l)=\frac{z_{\max}-z_0}{z_\infty-z_0}=\frac{z_\infty-z_0}{z_{\min}-z_0}\simeq\sqrt{\frac{z_{\max}-z_0}{z_{\min}-z_0}}=\frac{z_{\max}-z_\infty}{z_\infty-z_{\min}} \tag{11}$$

instead of Formulas (9) and (10).

In certain cases an interferometer in which the reflector is replaced by a second piezocrystal [10] is utilized instead of an ordinary interferometer. A voltage proportional to the amplitude B of the vibrations of the crystal boundary is measured across the electrodes of this receiving crystal.

Making use of the boundary conditions at piezocrystal-liquid and liquid-reflector boundaries, it is possible to derive an expression which relates B with the displacement amplitude A of the radiating surface:

$$B=A\frac{(1+\delta)e^{-ikl}}{1-\delta e^{-2ikl}},\ \text{etc.} \tag{12}$$

where $k = 2\pi/\lambda - i\alpha$. If A did not depend upon the mechanical load of the crystal (i.e., upon the position of the reflector) as Fry [10] assumed, then Formula (12) could be used directly for calculating α. However in actual fact, A depends upon the load both directly and in terms of the voltage across the crystal (this voltage varies when the current through the crystal varies). However, it is not difficult to express B directly in terms of the emf of the source V_0 and the parameters of the equivalent circuit. Thus for the circuit in Fig. 2 we have

$$\begin{aligned}B&=\frac{1}{i\omega}\frac{V_0R_0(1+\delta)/2\beta}{\left(R_0r_0+R_0r+r_0r+\frac{\delta S}{4\beta^2}\right)e^{ikl}-\delta\left(R_0r_0+R_0r+r_0r-\frac{3S}{4\beta^2}\right)e^{-ikl}}=\\&=\frac{a}{be^{ikl}+ce^{-ikl}},\end{aligned} \tag{13}$$

[i. e., a function which is of the same form as Equation (12)]. An analogous result can also be obtained for the circuit in Fig. 1. The maximums and minimums for the amplitude of vibration B, of the receiving crystal which correspond to the even and odd [depending upon the sign of the coefficient C in Equation (14)] values of $N = 4\ell/\lambda$ are equal to

$$B_{\max}=\frac{a}{be^{\alpha l}-|c|e^{-\alpha l}},$$
$$B_{\min}=\frac{a}{be^{\alpha l}+|c|e^{-\alpha l}}$$

---

* Mason derived only Formula (8); here he introduced the unnecessary (and not always correct) approximations $z \ll R_0$ and $1-\delta \ll 1$. In this formula, as well as in the analogous formula for a bridge circuit, he allowed a substantial error by incorrectly making use of the condition requiring the smallness of $1-\delta$.

Further, it is possible to compose the combinations:

$$1/B_{\max} + 1/B_{\min} = \frac{2b}{a} e^{\alpha l} \tag{14}$$

for

$$1/B_{\min} - 1/B_{\max} = \frac{2|c|}{a} e^{-\alpha l} \tag{15}$$

and the coefficient of absorption α can be determined from the slope of the graph representing the dependency of the logarithm of one of these quantities upon ℓ.

The formulas derived here for calculating the absorption coefficient are more convenient and more rigorous than those which have been utilized up to this time. It can be expected that the application of these formulas will permit a sufficiently reliable measurement of sound absorption in gases and in liquids which evidence an intense absorption. Of course, the interferometer cannot be used for the measurement of small values of absorption, since it is technically difficult to achieve large displacements of the reflector while retaining a sufficiently rigorous parallel relationship between the reflector and the radiating plate and a sufficiently uniform sound field.

The work described in this paper was carried out in the ultrasonic laboratory of the molecular physics staff of Leningrad University.

I express my appreciation to I. G. Mikhailov for his attention to this work and his valuable suggestions.

LITERATURE CITED

[1] P. E. Krashnushkin, "Theory of the ultrasonic interferometer," Doklady Akad. Nauk SSSR 27, 3, pp. 113-126 (1940). "Design of a Pierce interferometer," Sci. Record Moscow State Univ. 74, 73-86 (1944).

[2] J. W. F. Bell, "Satellite resonances in ultrasonic interferometer," Proc. Phys. Soc. (London) 63 B, 11, 958-964 (1950).

[3] J. C. Hubbard, "The acoustic resonator interferometer; I – The acoustic system and its equivalent network," Phys. Rev. 38, 5, 1011-1019 (1931). II – "Ultrasonic velocity and absorption in gases" 41, 4, 523-535 (1932), Errata, 46, 6, 525 (1934).

[4] L. Beliavskaia, "A new method for measuring the absorption coefficient of ultrasonic waves in gases," Bull. Acad. Sci. USSR, Division of Natural Sciences 7, 917-925 (1935).

[5] F. Fox, "Ultrasonic interferometer for liquid media," Phys. Rev. 52, 9, 973-981 (1937).

[6] U. P. Mason, Piezoelectric Crystals and Their Application in Ultrasonics (Foreign Lit. Press, Moscow, 1952).

[7] I. G. Mikhailov, Propagation of Ultrasonic Waves in Liquids (State Tech. Press, Moscow-Leningrad, 1949).

[8] K. F. Herzfeld, "Reflection of sound," Phys. Rev. 53, 11, 899-906 (1938).

[9] R. S. Alleman, "Dissipative acoustic reflection coefficients in gases by ultrasonic interferometry," Phys. Rev. 55, 1, 87-93 (1939).

[10] W. J. Fry, "The double crystal acoustic interferometer," J. Acoust. Soc. Am. 21, 1, 17-28 (1949).

Leningrad State University Received January 12, 1956

PROPAGATION OF ELASTIC WAVES IN A MEDIUM WITH CYLINDRICAL CHANNELS

V. V. Tiutekin

The paper develops the theory for the propagation of elastic waves in a medium with cylindrical channels. For the particular case of a rubber-like material an expression is obtained for the wave number for waves which are propagated in the direction of the channel axes under conditions where the radius of the channels is small in comparison to the wavelength of the shear wave. On the basis of the theoretical relationships a nomogram is plotted for the computation of the complex wave number according to the specified mechanical parameters of the rubber-like material and the geometric dimensions of the channels.

1. The problem of creating artificial acoustical absorbing media is of interest from the point of view of the principle involved and is of great practical significance. In particular, one of the possible types of artificial absorbing media is of interest — a rubber-like material with cylindrical channels.* A similar medium can be obtained by using spherical cavities [1]. In both of these cases the medium can be used in the capacity of a sound absorber. It would seem possible to accurately calculate the acoustical parameters of the micro-inhomogeneous media.** This paper is devoted to the theory of the propagation of elastic waves in a rubber-like material with cylindrical channels and to the determination of the acoustical parameters of such an artificial absorbing medium.

2. Let there be an unlimited isotropic medium whose properties can be described by a density ρ and the Lame elastic coefficients λ and μ. Cylindrical channels whose axes are parallel to one another penetrate the entire thickness of the medium. The channels are so arranged that a "dense packing" consisting of proper hexagons whose centers are the axes of the channels (Fig. 1) are formed in any plane which is perpendicular to the axes of the channels. Let us find the properties of such a micro-inhomogeneous medium if a plane sound wave is incident upon the medium in a direction parallel to the axes of the channels.

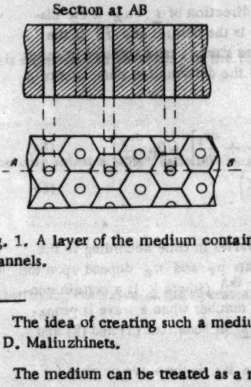


Fig. 1. A layer of the medium containing cylindrical channels.

It is evident that we can treat the entire medium as if it consisted of individual hexahedral prisms each of which contains a cylindrical channel which is infinitely long. Since symmetry obtains in this case, all of the prisms behave identically and the properties of the medium may be determined if the properties of one prism are known. When a wave is present which is being propagated in the direction of the channel axes there will be no normal displacements or tangential stresses at any of the boundaries of the prism. The boundary conditions which derive from such a disposition of the channel axes take into account the interaction between adjacent channels in the simplest manner. If we rejected a symmetrical disposition of the channels, then it would be necessary to examine the behavior of one channel in a limitless medium and then to take into account its interaction with adjacent channels. In order to simplify the calculation let us replace each of the prisms by a round hollow cylinder of infinite length and let us assume that the boundary conditions which obtain on each of the boundaries of the prism (Fig. 2) obtain on the external surface of the cylinder.

---

* The idea of creating such a medium and the approximate calculation of its elastic properties are due to G. D. Maliuzhinets.

** The medium can be treated as a micro-inhomogeneous one if the characteristic dimensions of the cavities are small in comparison to the wavelength of the sound wave in the medium.

Let us denote the outer radius of the cylinder by d and the inner radius of the cylinder (the radius of the channel) by a. We shall solve the problem in a cylindrical system of coordinates. The origin shall be placed at the axis of the cylinder and the coordinate z will be measured along the axis; then the coordinate r will specify the distance of a point from the axis of the cylinder. Since symmetry obtained here, there will be no dependence upon the angle θ.

In this system of coordinates the equations of motion can be written in the following manner [2]:

$$\rho \frac{\partial^2 u_r}{\partial t^2} = (\lambda + 2\mu) \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (r u_r)}{\partial r}\right] + (\lambda + \mu)\frac{\partial^2 u_z}{\partial r \partial z} + \mu \frac{\partial^2 u_r}{\partial z^2} \tag{1}$$

and

$$\rho \frac{\partial^2 u_z}{\partial t^2} = (\lambda + 2\mu)\frac{\partial^2 u_z}{\partial z^2} + \frac{\lambda + \mu}{r}\,\frac{\partial^2 (r u_r)}{\partial r\, \partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right). \tag{2}$$

The boundary conditions acquire the form:

$$\widehat{rz} = 0,\ \widehat{rr} = 0 \text{ for } \quad r = a, \qquad \text{(3a), (3b)}$$

$$u_r = 0,\ \widehat{rz} = 0 \text{ for } \quad r = b. \qquad \text{(4a), (4b)}$$

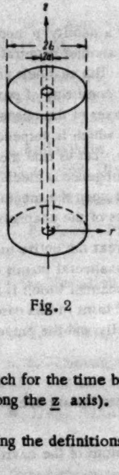


Fig. 2

Here u_r is the displacement in the direction of r, u_z is the displacement in the direction of z, $\widehat{rr}$ is the normal and $\widehat{rz}$ is the tangential stress in the medium; these stresses are expressed in terms of the parameters of the medium and the deformation components in the following manner [2]:

$$\widehat{rr} = \lambda\left[\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_z}{\partial z}\right] + 2\mu \frac{\partial u_z}{\partial r}, \tag{5a}$$

$$\widehat{rz} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right). \tag{5b}$$

Let us assume that the process varies in time according to the law $e^{-i\omega t}$, and that the displacements u_r and u_z depend upon the coordinate z according to the law e^{-ikz} (where k is a certain constant which for the time being is unknown and which plays the part of the wave number when a wave is propagated along the z axis). Substituting for the derivatives with respect to t and z in Equations (1) and (2) and introducing the definitions $k_l^2 = \frac{\rho\omega^2}{\lambda + 2\mu}$ and $k_t^2 = \frac{\rho\omega^2}{\mu}$, (where k_l is the wave number for longitudinal waves and k_t is the wave number for transverse waves in a solid medium), we rewrite Equations (1) and (2) in the following manner:

$$\frac{1}{k_l^2}\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru_r)}{dr}\right]+ik\left(\frac{1}{k_l^2}-\frac{1}{k_t^2}\right)\frac{du_z}{dr}+\left[1-\left(\frac{k}{k_t}\right)^2\right]u_r=0 \tag{6}$$

and

$$\frac{1}{k_l^2}\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right)+ik\left(\frac{1}{k_l^2}-\frac{1}{k_t^2}\right)\frac{A}{r}\frac{d}{dr}(ru_r)+\left[1-\left(\frac{k}{k_l}\right)^2\right]u_z=0. \tag{7}$$

In order to simplify this form we introduce a second-order differential operator D_r with respect to a certain quantity $\underline{u}$:

$$D_r u=\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru)\right).$$

Then it is evident that D_r^2 will be of the form:

$$D_r^2 u=\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rD_r u)\right)=\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right]\right]\right\}.$$

Introducing the definitions

$$\frac{du_z}{dr}=g,\ k_1^2=k_l^2-k^2,\ k_2^2=k_t^2-k^2 \tag{8}$$

and eliminating the quantity u_r from Equations (6) and (7), we obtain a fourth-order differential equation for the quantity $\underline{g}$:

$$D_r^2 g+(k_1^2+k_2^2)D_r g+k_1^2k_2^2 g=0. \tag{9}$$

This equation can be written in the following manner:

$$(D_r+k_1^2)(D_r+k_2^2)\,g=0.$$

The only solution which satisfies Equation (9) is the solution of the following system of equations:

$$\begin{aligned}D_r g+k_1^2 g&=0,\\ D_r g+k_2^2 g&=0.\end{aligned} \tag{10}$$

Substituting the value of the operator D_r into Equation (10), we obtain

$$\begin{aligned}\frac{d^2g}{dr^2}+\frac{1}{r}\frac{dg}{dr}+\left(k_1^2-\frac{1}{r^2}\right)g&=0,\\ \frac{d^2g}{dr^2}+\frac{1}{r}\frac{dg}{dr}+\left(k_2^2-\frac{1}{r^2}\right)g&=0.\end{aligned} \tag{10'}$$

A solution of this system of equations is the linear combination of first-order Bessel and Neuman functions of the arguments $k_1 r$ and $k_2 r$. Taking Equation (8) into account, we obtain a general solution for the quantity u_z in the form

$$u_z = C_1 J_0(k_1 r) + C_2 N_0(k_1 r) + C_3 J_0(k_2 r) + C_4 N_0(k_2 r), \tag{11}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. Substituting the value of u_z from Equation (11) into Equation (7), we find the solution for u_r:

$$u_r = iC_1 \frac{k_1}{k} J_1(k_1 r) + iC_2 \frac{k_1}{k} N_1(k_1 r) - iC_3 \frac{k}{k_2} J_1(k_2 r) - \\ - iC_4 \frac{k}{k_2} N_1(k_2 r). \tag{12}$$

The four arbitrary constants and the unknown quantity $\underline{k}$ (which plays the part of a wave number for waves which are propagated along the $\underline{z}$ axis) are included in Expressions (11) and (12) which have been obtained for u_z and u_r. In order to determine the four arbitrary constants we make use of the four boundary conditions – (3a), (3b), (4a) and (4b). After substituting the values of u_z and u_r from (11) and (12) into these expressions, we obtain a system which consists of four homogeneous equations with respect to C_1, C_2, C_3 and C_4:

$$C_1 \frac{k_1}{k} J_1(k_1 a) + C_2 \frac{k_1}{k} N_1(k_1 a) - C_3 \frac{k}{k_2}\left(1 - \frac{k_l^2}{2k^2}\right) J_1(k_2 a) - \\ - C_4 \frac{k}{k_2}\left(1 - \frac{k_l^2}{2k^2}\right) N_1(k_2 a) = 0, \tag{3'a}$$

$$C_1\left[\left(1 - \frac{k_l^2}{2k^2}\right) J_0(k_1 a) + \frac{k_1}{k^2 a} J_1(k_1 a)\right] + \\ + C_2\left[\left(1 - \frac{k_l^2}{2k^2}\right) N_0(k_1 a) + \frac{k_1}{k^2 a} N_1(k_1 a)\right] + \\ + C_3\left[J_0(k_2 a) - \frac{1}{k^2 a} J_1(k_2 a)\right] + C_4\left[N_0(k_2 a) - \frac{1}{k_2 a} N_1(k_2 a)\right] = 0, \tag{3'b}$$

$$C_1 \frac{k_1}{k} J_1(k_1 b) + C_2 \frac{k_1}{k} N_1(k_1 b) - C_3 \frac{k}{k_2} J_1(k_2 b) - C_4 \frac{k}{k_2} N_1(k_2 b) = 0, \tag{4'a}$$

$$C_1 \frac{k_1}{k} J_1(k_1 b) + C_2 \frac{k_1}{k} N_1(k_1 b) + C_3 \frac{k}{k_2}\left(1 - \frac{k_l^2}{2k^2}\right) J_1(k_2 b) - \\ - C_4 \frac{k}{k_2}\left(1 - \frac{k_l^2}{2k^2}\right) N_1(k_2 b) = 0. \tag{4'b}$$

A condition for nonzero solutions for the constants C_1, C_2, C_3 and C_4 is the condition which requires the determinant of this system to go to zero:

$$\left|\begin{matrix}
\frac{k_1}{k}J_1(k_1a) & \frac{k_1}{k}N_1(k_1a) \\
\frac{k_1}{k^2a}J_1(k_1a)+\left(1-\frac{k_t^2}{2k^2}\right)J_0(k_1a) & \frac{k_1}{k^2a}N_1(k_1a)+\left(1-\frac{k_t^2}{2k^2}\right)N_0(k_1a) \\
\frac{k_1}{k}J_1(k_1b) & \frac{k_1}{k}N_1(k_1b) \\
\frac{k_1}{k}J_1(k_1b) & \frac{k_1}{k}N_1(k_1b)
\end{matrix}\right. \rightarrow$$

$$\rightarrow \left.\begin{matrix}
\frac{k}{k_2}\left(1-\frac{k_t^2}{2k^2}\right)J_1(k_2a) & \frac{k}{k_2}\left(1-\frac{k_t^2}{2k^2}\right)N_1(k_2a) \\
\frac{1}{k_2a}J_1(k_2a)-J_0(k_2a) & \frac{1}{k_2a}N_1(k_2a)-N_0(k_2a) \\
\frac{k}{k_2}J_1(k_2b) & \frac{k}{k_2}N_1(k_2b) \\
\frac{k}{k_2}\left(1-\frac{k_t^2}{2k^2}\right)J_1(k_2b) & \frac{k}{k_2}\left(1-\frac{k_t^2}{2k^2}\right)N_1(k_2b)
\end{matrix}\right| = 0. \tag{13}$$

Let us introduce the following definitions:

$$\varepsilon = \frac{a}{b}; \quad \xi = \left(\frac{k}{k_l}\right)^2; \quad \alpha = \frac{1}{2}\frac{k_t^2}{k_l^2} = \frac{\lambda+2\mu}{2\mu};$$

$$x = k_2b = k_lb\sqrt{2\alpha-\xi}; \quad y = k_1b = k_lb\sqrt{1-\xi};$$

$$\varphi(u) = \varepsilon u\,\frac{J_1(u)\,N_0(\varepsilon u) - N_1(u)\,J_0(\varepsilon u)}{J_1(u)\,N_1(\varepsilon u) - N_1(u)\,J_1(\varepsilon u)},$$

where u is an arbitrary argument. In terms of these definitions Equation (13) can be written in the form of a transcendental equation with respect to the quantity ξ:

$$(1-\xi)\,[\xi\varphi(x)-\alpha] + (\alpha-\xi)^2\varphi(y) = 0. \tag{14}$$

This equation is the characteristic equation for determining the intrinsic values of the wave number k. It must be noted that an equation similar in form was obtained in [3] for the case of the propagation of axially-symmetrical elastic waves in a solid rod with a free surface.

3. Equation (14) is valid for a medium with any values of λ and μ. However it will henceforth be used only for rubber-like materials. As we know (see, for example, [4]), the latter materials, which have a volume elasticity which is approximately equal to the volume elasticity of water, have a shear modulus which is less by a factor of 10^2; here the loss coefficient for volume deformations is close to zero, while the loss coefficient for shear deformations attains values of the order of unity.* Such a relationship between these two

---

* The magnitude of the loss coefficient η is included in the expression for the complex shear modulus $\mu = \mu_0(1-i\eta)$, where μ_0 is the shear modulus without taking losses into account; the magnitude of the volume modulus can be considered to be purely real.

moduli permits us (by means of transforming volume deformations into shear deformations) to considerably diminish the effective compression modulus, and this leads to a decrease in the velocity of sound in the medium and to an increase of the loss effect. The transformation of one type of deformation into another type is accomplished by means of creating air cavities or channels in solid rubber. Here the relatively small volume deformations which arise when a longitudinal wave is incident upon the medium cause large shear deformations to appear which are concentrated in the vicinity of the surface of the cavity or channel. By varying the dimensions of the channel or the number of channels per unit of volume, it is possible to vary the degree to which the medium is "softened" and the magnitude of the absorption in it. Thus it is possible to obtain an artificial absorbing medium which is applicable for practical purposes.

In order to simplify the further calculations we shall somewhat transform Equation (14), making use of the above-discussed feature of the elastic properties of rubber. Since the value of $\alpha = \frac{\lambda + 2\mu}{2\mu}$ (its modulus) comprises a value of several hundred, and the value of ξ (as we shall show below) does not exceed several units, the argument x can be approximately written in the form

$$x \cong k_l b\sqrt{2\alpha} = k_t b.$$

The quantity x is a complex quantity, since when shear losses are present k_t is a complex quantity:*

$$\bar{k}_t = \frac{\omega}{c_t} = \frac{\omega\sqrt{\rho}}{\sqrt{\bar{\mu}}},$$

where $\bar{\mu} = \mu_0(1 - i\eta)$ is the complex shear modulus.

The evaluation of the function $\varphi(x)$ for $\eta = 1$, which is carried out by means of tables given in [5] and [6] showed that for all values of the argument which have a practical significance the magnitudes of the real and imaginary portions of the function $\varphi(\bar{x})$ do not exceed unity when the values of ϵ vary within the limits of from 0 to 0.5.** Since the quantity $\bar{\xi}$ has a modulus which is also close to unity, the terms $\bar{\xi}\varphi(\bar{x})$ and $\bar{\xi}$ can be neglected in comparison to $\bar{\alpha}$ in Equation (14); then, introducing the new function

$$\psi(\bar{y}) = \frac{J_1(\bar{y})N_1(\epsilon\bar{y}) - J_1(\epsilon\bar{y})N_1(\bar{y})}{J_1(\bar{y})N_0(\epsilon\bar{y}) - J_0(\epsilon\bar{y})N_1(\bar{y})},$$

it is possible to write Equation (14) in the form

$$\bar{y}\psi(\bar{y}) = \bar{\alpha}(k_l b)^2\epsilon. \tag{15}$$

The following physical analog corresponds to the wave number [Equation (15)]: a longitudinal wave is propagated inside a cylindrical tube with radius b and with infinitely rigid walls, along whose axis an elastic pipe with a radius a and with infinitely thin walls extends; the space between the walls of the tube and the walls of the pipe is filled with an ideal liquid having a volume compression modulus λ and an elasticity coefficient of $\bar{\gamma} = \frac{2\mu}{a}$, for the walls of the pipe (here $\bar{\gamma} = -\frac{p}{u_r}\Big|_{r=a}$, and p is the pressure in the liquid). It can be shown that Equation (15) is the characteristic equation for determining the wave number of the waves which are propagated in the liquid along the axis of the tube.

---

* Here, as henceforth, the line above the quantity denotes that it is complex.

** No calculation was made for $\epsilon > 0.5$.

4. Notwithstanding a certain simplification, Equation (15) is nevertheless an extremely complex transcendental equation; included in it, in addition to the unknown quantity $\bar{\xi}$, are a large number of parameters ($\bar{\alpha}$, ϵ, $k_{\ell} b$). Therefore we shall henceforth examine only certain particular cases of this equation. Let us examine the case where the values of the argument $\bar{y}$ are small. We expand the function $\psi(\bar{y})$ into a series and limit our analysis to the first term of the expansion; after appropriate transformation this function will be of the following form:

$$\psi(\bar{y}) = -\frac{\bar{y}(1-\varepsilon^2)}{2\varepsilon}. \tag{16}$$

We substitute its value into Equation (15) and obtain a solution for $\bar{\xi}$: *

$$\bar{\xi}_0 = \left(\frac{k}{k_l}\right)^2 = 1 + \frac{\lambda}{\bar{\mu}}\frac{\varepsilon^2}{1-\varepsilon^2}. \tag{17}$$

From this it follows that when $\epsilon = 0$ (a solid cylinder) $\xi = 1$ would be a solution of Equation (15) (i.e., $k = k_{\ell}$). Thus an ordinary plane sound wave is propagated in a solid rod with a radially reinforced external surface [this reinforcement characteristic corresponds to the conditions given by (4a) and (4b)]. When a very small cavity is created in such a rod (ϵ is small) the value of $\bar{\xi}_0$ will differ little from unity. Its difference from unity will be the larger, the smaller the shear modulus $\bar{\mu}$ of rubber in comparison to the volume elasticity modulus λ of rubber and the larger the magnitude of ϵ. Correspondingly, the propagation velocity of sound in the medium decreases and adsorption appears.

If we now take not one but two terms of the expansion obtained for the function $\psi(\bar{y})$ for the purposes of our calculation, then we obtain the magnitude of $\bar{\xi}$ in a second approximation. When two terms of the expansion are taken into account, the function $\psi(\bar{y})$ acquires the following value:

$$\psi(\bar{y}) = -\frac{\bar{y}}{2}\frac{1-\varepsilon^2}{\varepsilon(1+\bar{y}^2\ln\sqrt{\varepsilon})}. \tag{16*}$$

Substituting this value of $\psi(\bar{y})$ into the characteristic Equation (15), we obtain the following expression for $\bar{\xi}$:

$$\bar{\xi}_1 = \left(\frac{k}{k_l}\right)^2 = 1 + \frac{\lambda\varepsilon^2}{1-\varepsilon^2}\cdot\frac{1}{\bar{\mu} + \lambda\varepsilon^2\ln\sqrt{\varepsilon}\,(k_l b)^2} =$$
$$= 1 + \frac{\lambda\varepsilon^2}{1-\varepsilon^2}\cdot\frac{1}{\bar{\mu}\,[1+(\bar{k}_l a)^2\ln\sqrt{\varepsilon}]}. \tag{18}$$

Further, substituting $\bar{\mu} = \mu_0(1 - i\eta)$ and introducing the definitions

$$\mu' = \mu_0\,[1+(k_{t0}a)^2\ln\sqrt{\varepsilon}]\ \text{and}\ \eta' = \eta\frac{\mu_0}{\mu'} = \frac{\eta}{1+(k_{t0}a)^2\ln\sqrt{\varepsilon}}, \tag{19}$$

---

* In Equation (17), as everywhere from here on, we neglect the magnitude of 2μ in comparison to λ.

where $k_{t0} = \omega\sqrt{\frac{\bar{p}}{\mu_0}}$, we can write Expression (18) for $\bar{\xi}_1$ in the form

$$\bar{\xi}_1 = 1 + \frac{\lambda}{\bar{\mu}'}\frac{\varepsilon^2}{1-\varepsilon^2}, \tag{20}$$

where

$$\bar{\mu}' = \mu'(1 - i\eta').$$

From a comparison of Formulas (20) and (17) it is evident that the expression for the quantity $\bar{\xi}_1$ is analogous to the expression for the quantity $\bar{\xi}_0$; only instead of the true value of the shear modulus $\bar{\mu}$ a certain effective value $\bar{\mu}'$ appears in the expression for $\bar{\xi}_1$ Here the effective value of μ' decreases with an increase in frequency and the effective value of η' increases [this is true since $\ln\sqrt{\epsilon}$ is negative ($\epsilon < 1$)].

We shall now find the applicability limits for Solution (17). This solution was obtained on the basis of assuming that the second term in the expansion of the function $\psi(\bar{y})$ was small in comparison to the first term. From a comparison of Equations (16) and (16') it follows that this condition is fulfilled when $|\bar{y}^2 \ln\sqrt{\epsilon}| \ll 1$ (i.e., when $|k_t a| \ll \chi$); the value of χ which is determined by the expression $\chi = \frac{1}{\sqrt{\ln\sqrt{\varepsilon}}}$, is shown in Fig. 3 as a function of ϵ. From this figure it is evident that the magnitude of χ proves to be of the order of unity over an extremely wide range of values of ϵ. Therefore it is possible to utilize the relationship

$$|\bar{k}_t a| \ll 1, \tag{21}$$

as a criterion for the applicability of Formula (17) in calculating the wave number (i.e., the condition which specifies that the radius of the channel must be small in comparison to the wavelength of a shear wave in the rubber). Thus the solution which we have obtained can be considered to be a static approximation.

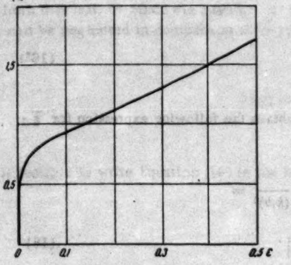


Fig. 3. Graph of the function $\chi(\epsilon)$.

Making use of Expression (17) and its applicability Criterion (21), it is possible to determine the values of the arbitrary constants C_1, C_2, C_3 and C_4 from the system of Equations (3'a), (3'b), (4'a) and (4'b) (more properly, the ratio of the first three of them to the fourth) and then, having substituted them into Equations (11) and (12), it is possible to find the solutions for the displacement components $\bar{u}_z$ and $\bar{u}_r$. After appropriate transformations and after introducing a new arbitrary constant C, we obtain

$$\bar{u}_z = Ce^{-i(\omega t - k_l\sqrt{\bar{\xi}_0} z)} \tag{22}$$

and

$$\bar{u}_r = iC \frac{\bar{\alpha}\varepsilon}{\sqrt{\bar{\xi}_0}} (k_l a) \left(\frac{b}{r} - \frac{r}{b}\right) e^{-i(\omega t - k_l \sqrt{\bar{\xi}_0} z)} .$$

Thus the displacement $\bar{u}_z$ does not depend upon $\underline{r}$ and is the displacement in a plane sound wave which is propagated along the $\underline{z}$ axis with a wave number $\bar{k} = k_{\ell} \sqrt{\bar{\xi}_0}$. The displacement $\bar{u}_r$ is maximum at the surface of the channel and decreases monotonically, going to zero at $r = b$. From Equation (22) it can easily be seen that in the general case the trajectory of motion for any point in the medium is an ellipse whose axis is inclined to the axis of the channel at an angle which is determined by the magnitude of the shear losses in the rubber.

5. It is possible to draw a certain analogy between the case of the propagation of elastic waves in a round cylindrical rod and the case of a propagation of elastic waves in a round pipe with a radially reinforced external surface. As we know, at relatively low frequencies (i.e., for wavelengths which are much greater than the diameter of the rod) the wave number has a value of $k = k_Y = \sqrt{\frac{\rho}{E}}\,\omega$, where E is Young's modulus for the material of the rod. In the case of a pipe the value of $\bar{k}$ [as can easily be seen from Equation (17)] is of the form $\bar{k} = \sqrt{\frac{\rho}{\bar{E}^*}}\,\omega$, for values of $|\bar{k}_t a|$ which are small in comparison to unity (the $\underline{s}$tatic case); here the effective modulus of the tube $\bar{E}^*$ is determined by the relationship

$$\frac{1}{\bar{E}^*} = \frac{1}{\lambda} + \frac{\varepsilon^2}{\bar{\mu}(1-\varepsilon^2)}. \qquad (23)$$

It is also easy to see that in this case the propagation velocity of the waves (both in the rod and in the pipe) does not depend upon the frequency (provided that the parameters of the material do not depend upon the frequency).

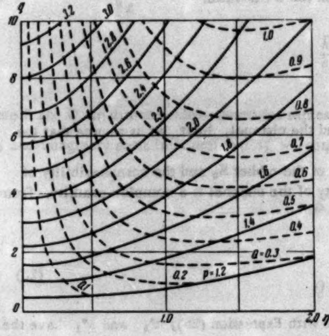


Fig. 4. Nomogram for determining the complex wave number $k = k_{\ell}(P + iQ)$ of the micro-inhomogeneous medium according to its known parameters (q and η).

In the other case, when some finite portion of the "Young's" wave will fit in the diameter of the rod, the so-called Raleigh Correction [7] is included in the expression for $\underline{k}$; it acquires the form $k = \sqrt{\frac{\rho}{E'}}\,\omega$, where E' is the effective Young's modulus which in this case is equal to

$$E' = E\left[1 - \frac{1}{2}\sigma^2 (k_Y a)^2\right]. \qquad (24)$$

Here σ is the Poisson coefficient, $\underline{a}$ is the radius of the rod. On the basis of comparing Equation (24) to Equation (19), it is evident that the Raleigh correction for the Young's modulus of the rod and the same correction for the Young's modulus of the pipe with a radially reinforced external surface both depend upon frequency in the same manner. Both of them are "dynamic" corrections to the static values of the wave numbers. Henceforth we shall also call the correction to the Young's modulus of the pipe a Raleigh correction.

6. In determining the complex wave number for the micro-inhomogeneous medium or for its element (the pipe) we examined the resultant effect produced by the solid rubber and the channel. The effect of the channel reduces, as we have already indicated, to the transformation of volume deformations in the solid rubber into shear deformations which are basically concentrated at the surface of the channel. However, it is possible to separate these two mechanisms if we introduce the compressibility of the rubber and the compressibility of the channel into our analysis separately. The expression for the velocity of longitudinal waves in the solid rubber can be written in the following manner;

$$c_l = \sqrt{\frac{1}{\rho S_0}},$$

where $S_0 = \frac{1}{\lambda}$ is the compressibility of the rubber. We shall assume that for a micro-inhomogeneous medium which consists of solid rubber with channels an analogous expression for $\bar{c}$ is valid:

$$\bar{c} = \sqrt{\frac{1}{\rho \bar{S}}},$$

where $\bar{S} = \frac{1}{\bar{E}^*}$ is the over-all compressibility of the rubber and the channel. Next, let us assume that the over-all compressibility $\bar{S}$ is a simple sum of the compressibility of the rubber S_0 and the compressibility of the channel $\bar{S}_1$ (i.e., that $\bar{S} = S_0 + \bar{S}_1$) and that the compressibility of the channel is a complex quantity. From Formula (23) it can easily be seen that in a static approximation

$$\bar{S}_1 = \frac{1}{\bar{\mu}} \frac{\varepsilon^2}{1-\varepsilon^2}. \tag{25}$$

Let us write $\bar{S}_1$ in the form $\bar{S}_1 = S_1' + i S_1''$ where [in accordance with Expression (25)] S_1' and S_1'' have the following values:

$$S_1' = \frac{\varepsilon^2}{(1-\varepsilon^2)(1+\eta^2)\mu_0} = S_0 \frac{q}{1+\eta^2}$$

and

$$S_1'' = \frac{\varepsilon^2 \eta}{(1-\varepsilon^2)(1+\eta^2)\mu_0} = S_0 \frac{q\eta}{1+\eta^2}, \tag{26}$$

and the parameter q is equal to $q = \frac{\lambda}{\mu_0} \frac{\varepsilon^2}{1-\varepsilon^2}$. It is interesting to note that $\frac{S_1''}{S_1'} = \eta$ and that this ratio does not depend upon the dimensions of the channel.

7. As we have indicated, the fundamental problem attacked by this paper is the investigation of the properties of a medium with cylindrical channels. This problem was reduced to the problem of the propagation of axially symmetrical waves along a pipe with a radially reinforced external surface; after that the wave number was found for such waves in a static approximation ($|\bar{k}_t a| \ll 1$). Now let us again pass over to an examination

of a limitless medium with cylindrical channels. Evidently the expression which has been found for the wave number for the case of one pipe will also be valid for the entire medium. An interesting conclusion follows from Equation (17) – the wave number of a micro-inhomogeneous medium with cylindrical channels does not depend upon the number of channels per unit of surface, but depends solely upon the quantity ϵ^2 which can be called the "perforation coefficient".

Let us examine Equation (17) for the wave number $\bar{k}$ in somewhat more detail; this expression can be written in the following form:

$$\bar{k} = k_l\sqrt{\bar{\xi}_0} = k_l\sqrt{1+\frac{q}{1-i\eta}}, \tag{27}$$

where q is determined by Expression (26). From (27) it is evident that the quantity $\frac{\bar{k}}{k_\ell}$ depends solely upon two parameters – q and η; this quantity can be calculated by plotting the corresponding nomogram. For this purpose we write $\frac{\bar{k}}{k_\ell}$ in the following manner:

$$\frac{\bar{k}}{k_l} = P + iQ,$$

where P and Q are dimensionless quantities representing the real and imaginary parts of $\frac{\bar{k}}{k_\ell}$. Separating the real and imaginary parts in Equation (27), we obtain

$$P = \rho_0\cos\frac{\theta}{2}$$

.and (28)

$$Q = \rho_0\sin\frac{\theta}{2},$$

where

$$\rho_0 = \sqrt[4]{1+q\frac{2+q}{1+\eta^2}}, \text{ and } \tan\theta = \frac{\eta q}{1+q+\eta^2}.$$

Figure 4 shows the nomogram for computing the magnitude of P and Q according to the known values of the parameters q and η. The nomogram is plotted on the basis of Equations (28). The values of the parameters q and η are plotted along the two axes, and the lines P = const and Q = const are drawn in this coordinate plane. Thus, knowing the elasticity parameters of rubber (λ, $\bar{\mu}$) and the perforation coefficient ϵ^2 it is possible to determine the static value of the wave number of an artificial absorbing medium which consists of rubber with cylindrical channels. Let us note that it is also possible to determine the wave number with the Raleigh Correction from the nomogram if for this case we use the quantity μ' from Equation (19) in the capacity of the shear modulus.

By means of the nomogram in Fig. 4 it is also possible to solve the reverse problem – determining the complex shear modulus $\bar{\mu}$ of rubber from the complex wave number (P + iQ) which is measured for an artificial absorbing medium by means of ordinary methods (provided the magnitudes of λ and ϵ^2 are known). Let us note that this can be done for various sonic or ultrasonic frequencies over a continuous frequency range, provided solely that the condition $|\bar{k}_t a| \ll 1$ is fulfilled over the specified range of frequencies.

The results of an experimental investigation of artificial absorbing media with cylindrical channels will be given in future papers.

In conclusion I express my deep appreciation to G. D. Maliuzhinets for his posing of the problem and supervision of the work, as well as to Iu. M. Sukharevskii for his valuable advice in carrying out this work.

LITERATURE CITED

[1] D. V. Sivukhin, "Diffraction of a plane sound wave at a spherical cavity," Acoust. J. 1, 1, 78-88 (1955).

[2] Liav, Mathematical Elasticity Theory (Foreign Lit. Press, Moscow, 1955).

[3] D. Bancroft, "The velocity of longitudinal waves in cylindrical bars," Phys. Rev. 59, 7, 588-593 (1941).

[4] T. Alfrey, Mechanical Properties of High-Order Polymers (Foreign Lit. Press, Moscow, 1954).

[5] Table of the Bessel Functions J_0 (z) and J_1 (z) for Complex Arguments (New York, 1943).

[6] Table of the Bessel Functions Y_0 (z) and Y_1 (z) for Complex Arguments (New York, 1951).

[7] Raleigh, Theory of Sound, Vol. 1 (Foreign Lit. Press, Moscow, 1954).

Acoustics Institute of the
Academy of Sciences, USSR, Moscow

Received December 12, 1955.

THE EFFECT OF THE PRESONIC FLOW VELOCITY UPON THE RADIATION IMPEDANCE OF A PISTON IN AN INFINITE BAFFLE

D. N. Chetaev

In [1] it was shown that the calculation of the acoustic impedance of a piston which is in an infinite baffle and which is radiating into a moving medium can be reduced to the evaluation of double integrals. In this paper computing formulas are provided in quadratures of tabulated functions. The substantial simplification of the computations has permitted an investigation of the dependence of the radiation impedance of a square piston upon the velocity of the presonic flow .

If the ideal compressibility of the liquid which fills the half-space $z > 0$ moves in the direction of the y axis with a velocity u that is less than the velocity of sound c ($u/c = \beta < 1$), and the piston S vibrates in an aperture cut out of an infinite baffle $z = 0$ according to the law $z = ve^{i\omega t}$, then the acoustic potential can be represented by the following integral [2]:

$$\varphi(P) = \frac{v}{2\pi} \iint\limits_{S'} \frac{e^{-ik'[r+\beta(y'-y_0')]}}{r} \, dS', \tag{1}$$

where $k' = k/\sqrt{1-\beta^2} \cdot (k = \omega/c)$ and $r = \sqrt{(x'-x_0')^2 + (y'-y_0')^2 + z_0'^2}$ is the distance from the point P (x_0, y_0, z_0) being examined in the field to the element of the piston in the coordinate system

$$x' = x, \; y' = y/\sqrt{1-\beta^2}, \; z' = z.$$

Let us note that Equation (1) for the velocity potential of the steady-state vibrations which are excited in the moving medium by the piston contained in the infinite baffle can be obtained (because of the fact that the baffle is a plane one) in the same manner [3] as the Raleigh formula — directly from the corresponding generalized Kirchoff theorem [4].

Making use of the linearity of the transformation of the coordinates x, y into the coordinates x', y', we express the acoustic impedance of the piston in terms of the new coordinates:

$$Z = \frac{i\omega\rho}{v} \iint\limits_{S} \varphi \, dS = \frac{i\omega\rho\sqrt{1-\beta^2}}{v} \iint\limits_{S'} \varphi \, dS', \tag{2}$$

where ρ is the density of the medium.

The radiation impedance is expressed by a four-fold integral. Representing the potential (1) at the points of the plate in terms of a curvilinear integral and a part of the plane wave (in the case of a stationary medium such a representation in terms of the sum of two discontinuous functions is possible for the potential at any point, [5, 6]), we reduce Expression (2) to a triple integral in a manner which is analogous to that utilized for the case of stationary medium [7, 8].

For the point P that lies on the piston itself, $r = \sqrt{(x' - x'_0)^2 + (y' - y'_0)^2}$, since $z'_0 = 0$. Let us introduce the polar coordinates r, θ with their center at the point P and with the polar axis directed parallel to the axis x'. If for the sake of simplicity we assume that the region occupied by the piston is convex when the equation of its contour $r = r(\theta)$ is given by a single-valued function for the condition $0 \leq \theta \leq 2\pi$, then we shall have the following expression at the points on the piston:

$$\varphi = \frac{v}{2\pi}\int_0^{2\pi}\int_0^{r(\theta)} e^{-ik'(1+\beta\sin\theta)r}\,dr\,d\theta,$$

since $dS' = dx'\,dy' = r\,dr\,d\theta$.

Integrating with respect to r, we obtain

$$\varphi = \frac{vi}{2\pi k'}\int_0^{2\pi}\left[e^{-ik'(1+\beta\sin\theta)\,r(\theta)} - 1\right]\frac{d\theta}{1+\beta\sin\theta} = -\frac{vi}{k} + \frac{vi}{2\pi k'}\int_0^{2\pi}\frac{\exp\left[-ik'(1+\beta\sin\theta)\,r(\theta)\right]}{1+\beta\sin\theta}\,d\theta,$$

since

$$\int_0^{2\pi}\frac{d\theta}{1+\beta\sin\theta} = \frac{2\pi}{\sqrt{1-\beta^2}}.$$

Substituting the resulting expression into Equation (2), we shall have

$$Z = c\rho S\left[1 - \frac{1-\beta^2}{2\pi S}\iint_{S'}\int_0^{2\pi}\frac{\exp\left[-ik'(1+\beta\sin\theta)\,r(\theta)\right]}{1+\beta\sin\theta}\,d\theta\,dS'\right].$$

Performing the similarity transformations $x' = ax'_1$ and $y' = ay'_1$, where a is some linear dimension of the region, we obtain

$$Z = c\rho S\left[1 - \frac{(1-\beta^2)\,a^2}{2\pi S}\iint_{S'_1}\int_0^{2\pi}\frac{\exp\left[-ik'a(1+\beta\sin\theta)\,r_1(\theta)\right]}{1+\beta\sin\theta}\,d\theta\,dS'_1\right]. \qquad (3)$$

Here the dimensionless factor a^2/S does not depend upon the magnitude of the area S, and the integration is conducted over the region S'_1 which is similar to S' and corresponds to the value a = 1. From this it is evident that the acoustic impedance can always be written as:

$$Z = c\rho S\left[1 - Y(ka) + iX(ka)\right],$$

where the form of the functions X (ka) and Y (ka) is determined solely by the shape of the piston and by the magnitude of the flow velocity.

In the inside integral of Expression (3) it is possible to use integration along an arc; then the expression for the radiation impedance will acquire the form:

$$Z = c\rho S\left[1-(1-\beta^2)\frac{a^2}{2\pi S}\times\right.$$
$$\left.\times\iint\limits_{S_1'}\oint\limits_C \frac{\exp\left[-ik'a(1+\beta\sin\theta)r_1(s)\right]}{(1+\beta\sin\theta)r_1(s)}\sqrt{1-(dr_1/ds)^2}\,ds\,dS_1'\right]. \tag{4}$$

From this it can be seen that Formula (4) also remains valid in the case when the piston comprises the algebraic sum of convex regions.

For the case of a rectangular piston it is possible to further simplify Formula (4).

Let us examine the piston $0 < x < a$, $0 < y < b$. If the side $\underline{a}$ is taken as the characteristic dimension, then the factor is equal to

$$a^2/2\pi S = 1/2\pi n \quad (n = b/a).$$

The region S'_1 in Formulas (3) and (4) will consist of the rectangle $0 < x'_1 < 1$, $0 < y'_1 < n' = n/\sqrt{1-\beta^2}$. (We have agreed to assume that the flow is directed along the $\underline{y}$ axis.) Henceforth we shall omit primes and indices in the coordinate notation.

The resistive and reactive portions of the dimensionless radiation impedance are equal to the following expressions according to Formula (4):

$$R = 1 - Y = 1 - \frac{1-\beta^2}{2\pi n}\int\limits_0^1\int\limits_0^{n'}\oint\limits_C \frac{\cos\left[k'a(1+\beta\sin\theta)r(s)\right]}{(1+\beta\sin\theta)r(s)}\sqrt{1-(dr/ds)^2}\,ds\,dy\,dx,$$

$$X = \frac{1-\beta^2}{2\pi n}\int\limits_0^1\int\limits_0^{n'}\oint\limits_C \frac{\sin\left[k'a(1+\beta\sin\theta)r(s)\right]}{(1+\beta\sin\theta)r(s)}\sqrt{1-(dr/ds)^2}\,ds\,dy\,dx.$$

On the left, right, lower and upper sides of the rectangle the distance $r(s)$ is respectively equal to

$$\sqrt{s^2+x^2},\quad \sqrt{s^2+(1-x)^2},\quad \sqrt{s^2+y^2},\quad \sqrt{s^2+(n'-y)^2},$$

where on each side $\underline{s}$ is measured in a counter-clockwise direction from the base of a perpendicular passing through P; for $\sin\theta$ we have respectively

$$-\frac{s}{r(s)},\quad \frac{s}{r(s)},\quad -\frac{y}{r(s)},\quad \frac{n'-y}{r(s)}.$$

Let us calculate X. We shall separate the inner integral into four parts which correspond to the four sides:

$$X = \frac{1-\beta^2}{2\pi n}(I_1 + I_2 + I_3 + I_4), \tag{5}$$

where

$$I_1 = \int_0^{n'}\int_0^{1}\int_{-(n'-y)}^{y} \frac{\sin k'a\left(\sqrt{s^2+x^2}-\beta s\right)}{\sqrt{s^2+x^2}-\beta s}\,\frac{x}{\sqrt{s^2+x^2}}\,ds\,dx\,dy,$$

$$I_2 = \int_0^{n'}\int_0^{1}\int_{-y}^{n'-y} \frac{\sin k'a\left(\sqrt{s^2+(1-x)^2}+\beta s\right)}{\sqrt{s^2+(1-x)^2}+\beta s}\,\frac{1-x}{\sqrt{s^2+(1-x)^2}}\,ds\,dx\,dy = I_1$$

(as can easily be seen if the substitution $1-x=\bar{x}$, $s=-s$ is made)

$$I_3 = \int_0^{1}\int_0^{n'}\int_{-x}^{1-x} \frac{\sin k'a\left(\sqrt{s^2+y^2}-\beta y\right)}{\sqrt{s^2+y^2}-\beta y}\,\frac{y}{\sqrt{s^2+y^2}}\,ds\,dy\,dx,$$

$$I_4 = \int_0^{1}\int_0^{n'}\int_{-(1-x)}^{x} \frac{\sin k'a\left[\sqrt{s^2+(n'-y)^2}+\beta(n'-y)\right]}{\sqrt{s^2+(n'-y)^2}+\beta(n'-y)}\,\frac{n'-y}{\sqrt{s^2+(n'-y)^2}}\,ds\,dy\,dx =$$

$$= \int_0^{1}\int_0^{n'}\int_{-x}^{1-x} \frac{\sin k'a\left(\sqrt{s^2+y^2}+\beta y\right)}{\sqrt{s^2+y^2}+\beta y}\,\frac{y}{\sqrt{s^2+y^2}}\,ds\,dy\,dx$$

(after making the substitutions of variables $n'-y=\bar{y}$, $s=-\bar{s}$).

We shall now calculate I_1, and for this purpose we shall change the order of integration and shall integrate with respect to x:

$$I_1 = \int_0^{n'}\int_0^{y}\int_0^{1} \frac{\sin k'a\left(\sqrt{s^2+x^2}-\beta s\right)}{\sqrt{s^2+x^2}-\beta s}\,\frac{x}{\sqrt{s^2+x^2}}\,dx\,ds\,dy +$$

$$+ \int_0^{n'}\int_0^{n'-y}\int_0^{1} \frac{\sin k'a\left(\sqrt{s^2+x^2}+\beta s\right)}{\sqrt{s^2+x^2}+\beta s}\,\frac{x}{\sqrt{s^2+x^2}}\,dx\,ds\,dy =$$

$$= \int_0^{n'}\int_0^{y}\int_{(1-\beta)s}^{\sqrt{s^2+1}-\beta s} \frac{\sin k'a\,\xi}{\xi}\,d\xi\,ds\,dy + \int_0^{n'}\int_0^{n'-y}\int_{(1+\beta)s}^{\sqrt{s^2+1}+\beta s} \frac{\sin k'a\,\xi}{\xi}\,d\xi\,ds\,dy =$$

$$= \int_0^{n'}\int_0^{y} \left[\operatorname{Si} k'a\left(\sqrt{s^2+1}-\beta s\right) - \operatorname{Si} k'a(1-\beta)s\right] ds\,dy +$$

$$+ \int_0^{n'}\int_0^{n'-y} \left[\operatorname{Si} k'a\left(\sqrt{s^2+1}+\beta s\right) - \operatorname{Si} k'a(1+\beta)s\right] ds\,dy.$$

In the integrals we have obtained we shall change the order of integration over the regions in the sy plane and shall integrate with respect to y (the expressions in the intergrands do not depend upon y):

$$I_1 = \int_0^{n'} [\mathrm{Si}\, k'a(\sqrt{s^2+1}+\beta s) + \mathrm{Si}\, k'a(\sqrt{s^2+1}-\beta s) -$$

$$- \mathrm{Si}\, k'as(1+\beta) - \mathrm{Si}\, k'as(1-\beta)]\,(n'-s)\,ds.$$

After making a simple substitution we finally obtain

$$I_1 + I_2 = \frac{2n^2}{1-\beta^2}\int_0^1 \Big[\mathrm{Si}\,\frac{kan}{1-\beta^2}\Big(\sqrt{s^2+\frac{1-\beta^2}{n^2}}+\beta s\Big) +$$

$$+ \mathrm{Si}\,\frac{kan}{1-\beta^2}\Big(\sqrt{s^2+\frac{1-\beta^2}{n^2}}-\beta s\Big) - \mathrm{Si}\,\frac{kans}{1+\beta} - \mathrm{Si}\,\frac{kans}{1-\beta}\Big](1-s)\,ds. \qquad (6)$$

In calculating the integrals I_3 and I_4 care is required in selecting the branches of the double-valued functions which make the transformation of variables possible. In [1] such care was not taken, and as a result the final formula given there is incorrect; that formula should be replaced by Formulas (5 - 7) of this present paper.

Evaluating

$$I_4 = \int_0^1 \int_{-x}^{1-x} \int_0^{n'} \frac{\sin k'a(\sqrt{s^2+y^2}+\beta y)}{\sqrt{s^2+y^2}+\beta y}\,\frac{y\,dy}{\sqrt{s^2+y^2}}\,ds\,dx =$$

$$= \int_0^1 \int_{-x}^{1-x} \int_{|s|}^{\sqrt{s^2+n'^2}+\beta n'} \Big(\frac{1}{1-\beta^2}\,\frac{\sin k'a\xi}{\xi} - \frac{\beta}{1-\beta^2}\,\frac{\sin k'a\xi}{\sqrt{\xi^2-s^2(1-\beta^2)}}\Big)\,d\xi\,ds\,dx,$$

it should be taken into account that when the substitution

$$\sqrt{s^2+y^2}+\beta y = \xi$$

has been made the inverse transformation is accomplished by means of the branch

$$y = \frac{-\beta\xi+\sqrt{\xi^2-s^2(1-\beta^2)}}{1-\beta^2}.$$

In calculating I_3 we assumed that

$$\sqrt{s^2+y^2}-\beta y = \xi;$$

then for $0 \leqslant y \leqslant |s|\beta/\sqrt{1-\beta^2}$, $y = \dfrac{\beta\xi-\sqrt{\xi^2-s^2(1-\beta^2)}}{1-\beta^2}$, and for

$$y \geqslant |s|\beta/\sqrt{1-\beta^2},\ y = \frac{\beta\xi+\sqrt{\xi^2-s^2(1-\beta^2)}}{1-\beta^2}.$$

On the basis of this,

$$I_3=\int_0^1\int_{-x}^{1-x}\int_0^{n'}\frac{\sin k'a\left(\sqrt{s^2+y^2}-\beta y\right)}{\sqrt{s^2+y^2}-\beta y}\,\frac{y\,dy}{\sqrt{s^2+y^2}}\,ds\,dx=$$

$$=\int_0^1\int_{-x}^{1-x}\int_{|s|}^{|s|\sqrt{1-\beta^2}}\left(\frac{1}{1-\beta^2}\,\frac{\sin k'a\xi}{\xi}-\frac{\beta}{1-\beta^2}\,\frac{\sin k'a\xi}{\sqrt{\xi^2-s^2(1-\beta^2)}}\right)d\xi\,ds\,dx+$$

$$+\int_0^1\int_{-x}^{1-x}\int_{|s|\sqrt{1-\beta^2}}^{\sqrt{s^2+n'^2}-\beta n'}\left(\frac{1}{1-\beta^2}\,\frac{\sin k'a\xi}{\xi}+\frac{\beta}{1-\beta^2}\,\frac{\sin k'a\xi}{\sqrt{\xi^2-s^2(1-\beta^2)}}\right)d\xi\,ds\,dx.$$

Thus,

$$I_3+I_4=\frac{1}{1-\beta^2}\int_0^1\int_{-x}^{1-x}\left\{\int_{|s|}^{\sqrt{s^2+n'^2}+\beta n'}+\int_{|s|}^{\sqrt{s^2+n'^2}-\beta n'}\right\}\frac{\sin k'a\xi}{\xi}\,d\xi\,ds\,dx+$$

$$+\frac{\beta}{1-\beta^2}\int_0^1\int_{-x}^{1-x}\left\{2\int_{|s|\sqrt{1-\beta^2}}^{|s|}-\int_{\sqrt{s^2+n'^2}-\beta n'}^{\sqrt{s^2+n'^2}+\beta n'}\right\}\frac{\sin k'a\xi}{\sqrt{\xi^2-s^2(1-\beta^2)}}\,d\xi\,ds\,dx.$$

The first term is transformed in a manner analogous to that used for the sum $I_1 + I_2$; in the second term, having changed the order of integration over the region sx, it is possible to integrate with respect to x:

$$I_3+I_4=\frac{2}{1-\beta^2}\int_0^1\Big[\operatorname{Si}\frac{ka}{1-\beta^2}\left(\sqrt{s^2(1-\beta^2)+n^2}+\beta n\right)+$$

$$+\operatorname{Si}\frac{ka}{1-\beta^2}\left(\sqrt{s^2(1-\beta^2)+n^2}-\beta n\right)-2\operatorname{Si}\frac{kas}{\sqrt{1-\beta^2}}\Big](1-s)\,ds+I, \qquad (7)$$

where

$$I=\frac{4\beta}{1-\beta^2}\int_0^1\int_s^{s/\sqrt{1-\beta^2}}\frac{(1-s)\sin ka\xi}{\sqrt{\xi^2-s^2}}\,d\xi\,ds-\frac{2\beta}{1-\beta^2}\int_0^1\int_{A_-(s)}^{A_+(s)}\frac{(1-s)\sin ka\xi}{\sqrt{\xi^2-s^2}}\,d\xi\,ds,$$

$$A_\pm(s)=\frac{1}{1-\beta^2}\left(\sqrt{s^2(1-\beta^2)+n^2}\pm\beta n\right).$$

Formulas (5), (6) and (7) yield the expression for X (ka) in terms of quadratures of the tabulated functions and the double integral I. The expression for Y (ka) is obtained in similar form, but the integral cosine replaces Si η and the cosine replaces the sine in the double integrals.

A portion of the integrals included in Formulas (6) and (7) and in the analogous expressions Y (ka) can be expressed in terms of tabulated functions:

$$2\int\limits_0^1 (1-s)\,\mathrm{Si}\,\alpha s\,ds = \mathrm{Si}\,\alpha + \frac{\cos\alpha - 2}{\alpha} + \frac{\sin\alpha}{\alpha^2},$$

$$2\int\limits_0^1 (1-s)\,\mathrm{Ci}\,\alpha s\,ds = \mathrm{Ci}\,\alpha - \frac{\sin\alpha}{\alpha} + \frac{\cos\alpha - 1}{\alpha^2}.$$

The remaining integrals must be tabulated.

Let us go on to the evaluation of the double integral. Changing the order of integration, we obtain

$$\int\limits_0^1 \int\limits_s^{s/\sqrt{1-\beta^2}} f\,d\xi\,ds = \left\{ \int\limits_1^{1/\sqrt{1-\beta^2}} \int\limits_0^1 + \int\limits_0^1\int\limits_0^{\xi} - \int\limits_0^{1/\sqrt{1-\beta^2}} \int\limits_0^{\xi\sqrt{1-\beta^2}} \right\} f\,ds\,d\xi,$$

$$\int\limits_0^1 \int\limits_{A_-(s)}^{A_-(s)} f\,d\xi\,ds = \left\{ \int\limits_{B_-}^{B_+}\int\limits_0^1 + \int\limits_{n/(1+\beta)}^{B_-} \int\limits_0^{\sqrt{\xi^2-(\xi\beta-n)^2}} - \int\limits_{n/(1-\beta)}^{B_+} \int\limits_0^{\sqrt{\xi^2-(\xi\beta+n)^2}} \right\} f\,ds\,d\xi.$$

$$B_\pm = \frac{1}{1-\beta^2}\left(\sqrt{1-\beta^2+n^2} \pm \beta n\right).$$

The integral with respect to s is taken as

$$\int \frac{(1-s)\,ds}{\sqrt{\xi^2-s^2}} = \arcsin\frac{s}{\xi} + \sqrt{\xi^2-s^2} + \text{const.}$$

Thus we obtain the quadratic formula:

$$\begin{aligned} I = {} & \frac{4\beta}{1-\beta^2} \int\limits_1^{1/\sqrt{1-\beta^2}} \sin ka\xi \left[\arcsin\frac{1}{\xi} + \sqrt{\xi^2-1} - \xi\right] d\xi + \\ & + \frac{4\beta}{1-\beta^2} \int\limits_0^1 \sin ka\xi\,[\pi/2 - \xi]\,d\xi - \\ & - \frac{4\beta}{1-\beta^2} \int\limits_0^{1/\sqrt{1-\beta^2}} \sin ka\xi\,[\arcsin\sqrt{1-\beta^2} - \xi(1-\beta)]\,d\xi - \\ & - \frac{2\beta}{1-\beta^2} \int\limits_{B_-}^{B_+} \sin ka\xi \left[\arcsin\frac{1}{\xi} + \sqrt{\xi^2-1} - \xi\right] d\xi - \\ & - \frac{2\beta}{1-\beta^2} \int\limits_{n/(1+\beta)}^{B_-} \sin ka\xi \left[\arcsin\sqrt{1-\left(\beta-\frac{n}{\xi}\right)^2} + |\beta\xi - n| - \xi\right] d\xi + \\ & + \frac{2\beta}{1-\beta^2} \int\limits_{n/(1-\beta)}^{B_+} \sin ka\xi \left[\arcsin\sqrt{1-\left(\beta+\frac{n}{\xi}\right)^2} + |\beta\xi + n| - \xi\right] d\xi. \end{aligned} \tag{8}$$

The double integral which is included in the expression for Y (ka) is of a completely similar form except for the fact that sin ka ξ is replaced by cos ka ξ.

TABLE 1

$X\,(kc;\ \beta)$

kc \ β	0.0	0.2	0.4	0.6	0.8
0.25	0.206	0.207	0.213	0.224	0.242
0.50	0.391	0.393	0.401	0.400	0.370
0.75	0.538	0.537	0.530	0.504	0.441
1.00	0.634	0.627	0.601	0.545	0.490
1.25	0.674	0.660	0.624	0.550	0.512
1.50	0.661	0.642	0.589	0.540	0.512
1.75	0.599	0.584	0.538	0.521	0.504
2.00	0.517	0.501	0.477	0.490	0.483
2.25	0.415	0.409	0.418	0.446	0.454
2.50	0.316	0.323	0.361	0.394	0.427
2.75	0.231	0.252	0.313	0.346	0.395
3.00	0.168	0.200	0.268	0.305	0.355

TABLE 2

$R\,(kc;\ \beta)$

kc \ β	0.0	0.2	0.4	0.6	0.8
0.25	0.031	0.032	0.036	0.047	0.075
0.50	0.120	0.124	0.138	0.166	0.210
0.75	0.255	0.262	0.286	0.321	0.327
1.00	0.420	0.429	0.452	0.468	0.440
1.25	0.595	0.601	0.611	0.589	0.550
1.50	0.762	0.761	0.745	0.688	0.639
1.75	0.902	0.892	0.848	0.775	0.721
2.00	1.007	0.986	0.910	0.853	0.792
2.25	1.070	1.041	0.967	0.918	0.847
2.50	1.093	1.062	0.999	0.962	0.895
2.75	1.084	1.058	1.018	0.987	0.934
3.00	1.053	1.038	1.029	0.997	0.962

And thus, Formulas (5 - 8) reduce the calculation of the effect of the presonic flow velocity upon the radiation impedance of a rectangular piston in an infinite baffle to quadratures. The study of this effect can be of interest from the point of view of the possibility of utilizing it (in principle) for determining the flow velocity. The results of computing the acoustic impedance of a square piston for the values $\beta = 0.2, 0.4, 0.6$ and 0.8 are given in Tables 1 (reactive component) and 2 (resistive component). These tables also list the corresponding values for a stationary ($\beta = 0$) square piston; these values were calculated by a method analogous to the one described above [7, 8] and (later) by other methods [9, 10, 11]. Here the argument is kc, where c is the radius of a circle which has an area equal to the area of the square.

The author expresses his appreciation to N. N. Andreev and L. M. Brekhovskikh for their interest in this work, which served as a stimulus in carrying out the calculations; I also wish to thank A. N. Tikhonov for various remarks which he made.

LITERATURE CITED

[1] D. N. Chetaev, "The acoustic impedance of a moving plane emitter," Doklady Akad. Nauk SSSR 90, - 3, 355-358 (1953).

[2] S. S. Voit, "Propagation of waves from a sound emitting disc in a moving medium," Appl. Math. and Mechanics (USSR) 16, 6, 699-705. (1952).

[3] G. Lamb, Hydrodynamics, § 290 (State Tech. Press, Moscow-Leningrad, 1947).

[4] G. I. Blokhintsev, The Acoustics of An Inhomogeneous Moving Medium, (State Tech. Press, Moscow-Leningrad, 1946).

[5] A. Schoch, Betrachtungen über das Schallfeld einer Kolbenmembran, Akust. Z. 6, 6, 318-326 (1941).

[6] D. N. Chetaev, "The radiation of sound by a piston," Doklady Akad. Nauk SSSR 76, 6, 813-816 (1951).

[7] D. N. Chetaev, "Concerning a boundary problem of a wave equation," Progr. Math. Sci. (USSR) 5, 6, 136-137 (1950).

[8] D. N. Chetaev, "The impedance of a rectangular plate which vibrates in an area cut out of a plane wall," Appl. Math. and Mechanics (USSR) 13, 439-444 (1951).

[9] Y. Nomura and Y. Aida, "On the radiation impedance of a rectangular plate with an infinitely large fixed baffle," Sci. Rep. Res. Inst. Tohoku Univ. (Japan) B, 1, 2, 337-347 (1951).

[10] G. W. Swenson and W. E. Johnson, "Radiation impedance of a rigid square piston in an infinite baffle," J. Acoust. Soc. Am. 24, 1, 84 (1952).

[11] H. Stenzel, Die akustische Strahlung der rechteckigen Kolbenmembran, Acustica 2, 6, 261-281 (1952).

Geophysics Institute of the
Academy of Sciences, USSR

Received September 14, 1955.

THE SIGNIFICANCE OF TEMPORAL FACTORS IN THE RECEPTION OF COMPLEX SOUNDS

L. A. Chistovich

The paper examines the human discrimination of a complex signal when complex interference is present. The paper demonstrates the prime significance of temporal factors in the process of discriminating between the useful signals and the interference.

Under natural conditions of human activity conditioned sound signals are, as a rule, complex sounds. In order for the sound analyzer to "measure" a complex sound signal (i.e., to determine its timbre, loudness, spatial localization etc.) this signal must evidently first of all be segregated as an individual independent signal from the mass of complex sound stimuli which usually act simultaneously with it. The problem arises as to what mechanisms and what laws govern the segregation of individual complex signals from that over-all complex perturbation which exists at each moment in time at the input to the auditory canal and is composed of the whole mass of noise and sound (speech, music, industrial noise etc.) which simultaneously act upon a human being.

It is evident that such a segregation of signals cannot, in principle, be achieved by the peripheral frequency analysis which is used (in accordance with modern concepts concerning the operation of the auditory system) to transform the spectrum of the perturbations into a spatially distributed pattern of excitation and inhibition in the neurons of the auditory paths [1]. In order that this segregation might take place it is necessary that there be certain supplementary mechanisms which could unite the excitation of one group of neurons into a single system (figuratively speaking, would cause the excitation of this group to appear in one channel), and could unite the excitation which simultaneously existed in other groups into another system (i.e., this excitation would appear in another channel).

The first stage of studying this proposed mechanism must consist of a clarification of those conditions for which complex sounds are received as one signal, and of those conditions for which these sounds are decomposed into individual signals. Certain facts which derive from the experience of practical life permit us to make the assumption that an essential condition for the combination of sounds (or tonal components) into one signal is the simultaneity of their various intensities.

Thus it is well known that we receive sounds from one source as a single signal (speech sounds enunciated by a human being, the cry of an animal, etc.) when the components of that sound vary synchronously due to the very conditions under which the sound is produced. On the contrary, we receive sounds from different sources (these sounds vary independently of one another in time because the sources are independent) as separate signals. The purpose of the experiments which are discussed in this paper was the verification of this postulate [i.e., the investigation of the difference in the reception of complex sounds under conditions where they were synchronously (simultaneously) cut in and under conditions where one sound is associated with another after a definite time interval]. Experiments were conducted with complex sounds having a discrete spectrum (these sounds approximated vowel sounds) and with noise signals (two noise bands).

In the first series of experiments the subject was given the problem of segregating a complex signal previously known to him from an aggregate perturbation. In the second series of experiments the subject was given the problem of determining the nature of the sound (timbre) of the aggregate perturbation.

In the first series of experiments two groups of harmonics having a fundamental frequency of 200 cps, or two noise bands were used in the capacity of the complex signals. In order to obtain the group of harmonics a "spike" pulse generator * was used; this generator was connected to two lines consisting of band filters and attenuators. The pulse generator consisted of a nonlinear amplifier (connected in the form of a trigger circuit) and a differentiating circuit. The two output circuits of the generator were separated by means of cathode followers. The repetition frequency of the pulses was set by means of an audio oscillator of the GZ-2 type. The spectrum of the harmonic groups which we utilized is shown in Fig. 1.

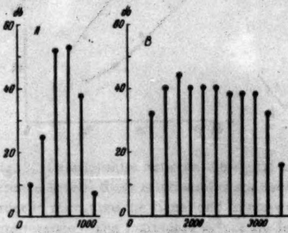


Fig. 1. Spectra of the signal A and the interference B.

In order to obtain two noise signals we used a noise generator of the LIG-24 type. By means of band filters we separated out two frequency bands from the noise spectrum (500-1000 cps and 1500-3200 cps); the noise level in these bands was controlled by means of attenuators. The voltage from both lines was fed to the same electrodynamic telephone. In order to guarantee a normal output load for the attenuators additional resistors were connected into the telephone circuit.

In the second series of experiments complex sounds were utilized which had a discrete spectrum and which were very close to the sounds of the vowels ɛ̄ and ä. The first of them (the sound which was similar to the vowel ɛ̄) consisted of a group of harmonics having a fundamental frequency of 200 cps whose intensity diminished and whose frequencies were distributed in the band 3,000-8,000 cps. The second sound (the sound similar to the vowel ä) consisted of a 1000 cps tone which was amplitude modulated at 200 cps. In order to obtain such a signal we utilized a ring modulator which was so designed that it was possible to pass the carrier frequency to the output and to control the carrier frequency amplitude. The depth of modulation was controlled by means of an oscilloscope and was established at about 100%. Both of the complex signals were fed (each one through its attenuator) to one common telephone. The experiments were run on 12 subjects with normal hearing.

The purpose of the first series of experiments was a comparison of the possibilities for segregating the complex signal under conditions when it was simultaneously cut in with another complex signal (we shall agree to call the latter the "interference") and under conditions when these signals are cut in at different times (the interference is cut in somewhat earlier, and then the "signal" is combined with it).

In order to quantitatively characterize the possibility of segregating the signal under the conditions which are indicated above we made use of a method which involved determining the curves for the relationship between the interference intensity level and the signal intensity level; these curves are required for signal recognition. Both the signal and the interference were complex sounds and consisted either of groups of harmonics with the same fundamental frequency or of noise bands. The signal was always lower in frequency than the interference (See Fig. 1).

The experiment was posed in the following manner: we explained to the subject that he must segregate a definite signal contained in a complex sound. This signal had to be memorized, and as soon as the observer detected and recognized it he was to inform us of it. The explanation was clarified by the presentation of the sounds. Next, the signal and interference recognition thresholds were determined, and the experiment was begun. In order to simplify the problem the stimuli were applied in pairs: at first the signal was cut in for 0.7-1.0 sec, and then after an interval of 0.6-0.8 sec the signal and interference were simultaneously cut in (for 0.7-1.0 sec).

---

* Developed by Engineer Soroko.

In other experiments the signal and interference were not cut in simultaneously (the interference was cut in 0.6 sec earlier). The duration of the signal was 0.7-1.0 sec. The level of the interference in these experiments was varied within the limits of from 10 to 80 db above the threshold. At each specified interference level we determined that minimum signal level at which the subject answered that he recognized the signal in seven cases out of ten. Data was obtained for seven subjects with normal hearing, of whom four had been trained in auditory measurements over a long period of time.

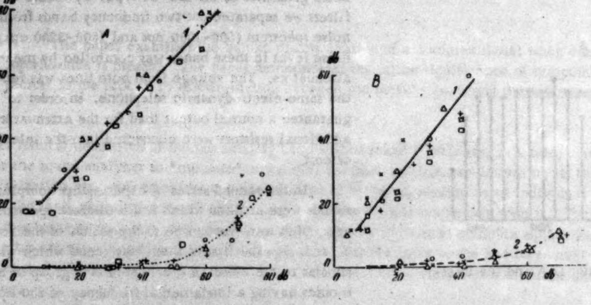


Fig. 2. Dependence of the signal recognition threshold upon the intensity level of the interference. Data for six subjects. 1) Signal and interference cut in simultaneously, 2) interference is cut in earlier than the signal. The interference intensity level is plotted along the axis of abscissas and the signal intensity level is plotted along the axis of ordinates; both quantities are plotted in db from the recognition threshold for complete silence. A) signal and interference consisting of a group of harmonics, B) signal and interference consisting of a noise band.

The results of the experiments are shown in Fig. 2, A and B. From these results it follows that the relationships between the signal and interference levels (the relationships necessary for recognizing the signal) proved to be completely different in the case where the signal and the interference are cut in simultaneously (the solid curve), and in the case where there is a time lag between the instants at which the signal and the interference are cut in (the dotted curve). In the case when the signal and the interference are cut in simultaneously it is necessary that the signal intensity level exceed the interference level in order for the signal to be recognized; in the case when there is a time lag the signal is recognized when its intensity level is considerably lower than the interference intensity level.

The report of the test subject was entirely different for these two cases. When the signal and the interference were cut in simultaneously the test subject reported that he had recognized the signal and explained that he heard one sound which was very similar in character to the sound of the signal. In the case when there was a time lag the test subject stated that there were two sounds — one sound was extraneous and the signal appeared against a background of this extraneous sound. The nature of the responses of the test subjects permitted us to assume that the different shapes of the masking curves in the cases of simultaneous and retarded application of the signal is caused by the fact that when the two stimuli are cut in simultaneously they do not decompose into a signal and interference; thus the acoustic stimulus is received as one sound which approximates first the sound of the signal, then the sound of the interference. It is evident that if this assumption is correct, then when it becomes possible to recognize the signal (when the signal is cut in simultaneously with the interference) this ability must be accompanied by the loss of an ability to recognize the interference. In order to verify this postulate we conducted experiments in which we determined the interference recognition threshold when the interference was simultaneously cut in with the signal (Fig. 3, Curve 2). From this figure it follows that it is a fact

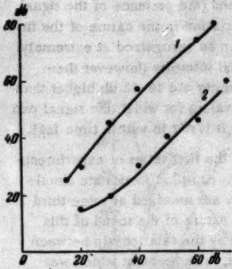


Fig. 3. Relationships between the signal and interference levels; these relationships are necessary in order to recognize the signal and to recognize the interference. The individual values are for one person. The case depicted is that of the simultaneous application of the signal and interference. Signal: 1-6 harmonics of 200 cps, interference: 7-17 harmonics of 200 cps. The interference intensity level is plotted along the axis of abscissas, the signal intensity level is plotted along the axis of ordinates; both quantities are plotted in db above the recognition threshold in complete silence.

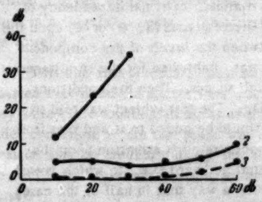


Fig. 4. Dependence of the signal recognition threshold upon the interference intensity level. The individual values are for one person. Signal: a noise band in the region 500-1000 cps, interference: a noise band in the region 1500-3200 cps; 1 and 2) the signal and noise are applied simultaneously, 3) the interference is cut in earlier than the signal. The interference intensity level is plotted along the axis of abscissas and the signal intensity level is plotted along the axis of ordinates.

that when the relationships between the signal and interference intensity levels were such that the test subject recognized the signal (Curve 1), the subject could not recognize the interference. In order to recognize the interference it was necessary that the intensity level of the interference exceed the signal intensity level (Curve 2).

The impossibility of decomposing the complex sound into a signal and interference when they were simultaneously cut in was characteristic for six of the seven test subjects; only one of the subjects could carry out such a task after prolonged training. The results of the experiments on this subject are shown in Fig. 4. Curve 1 shows that at first this test subject, as the remaining subjects, recognized the signal (a low noise band) only when its intensity level was greater than the interference intensity level. However thereafter she began to segregate the signal almost as well when it was simultaneously cut in (Curve 2) as when it was cut in with a time lag (Curve 3).

Describing the method which enabled her to do this, the test subject stated that she attempted to prolong the sound of the signal in her mind after it had been cut out; she did this in such a way as to make it seem that it continued up to the instant when it was again cut in together with the interference. This description coincides extremely closely with the description of the device proposed by Helmholtz [2] for segregating individual components contained in complex sounds. It should be noted that attempts by the other test subjects to apply this method were not crowned with success.

In all of the experiments discussed above we required that the test subjects give us answers concerning the signal only when they detected the signal as such (i.e., only when they heard a sound which corresponded to the signal in terms of its properties). It was quite obvious that the extremely great masking effect in the case when signal and interference were cut in simultaneously was caused by an inability to distinguish the signal from the interference, but not at all by the inability to detect its presence. In order to determine the detection threshold with respect to the presence of the signal under conditions of simultaneous application of signal and interference we utilized the following experimental procedure: the test subject was instructed to compare the interference with a complex sound consisting of the signal and various types of interference. The stimuli were applied in pairs – at first the interference was cut in, and then after an interruption of 0.4-0.6 seconds the signal and the interference were cut in simultaneously. The signal intensity level was diminished until the test subject stated that both sounds were completely identical. The results of these experiments are shown in Fig. 5 (Curve 3). It follows from this figure that the presence of the signal in the complex sound (the presence of the signal is registered as a variation in the nature of the interference sound) can be recognized at extremely small values of signal intensity (however these values of signal intensity are 10-15 db higher than the signal intensity values for which the signal can be recognized when it is cut in with a time lag).

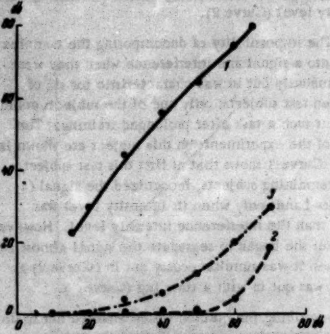


Fig. 5. Relationships which are necessary between the signal and interference levels in order to recognize the signal (1) and to detect its presence (3) under conditions when the signal and the interference are cut in simultaneously. The individual values were obtained for one person. The interference (2) is cut in earlier than the signal. Signal: 1-6 harmonics of 200 cps; interference: 7-17 harmonics of 200 cps. The interference intensity level is plotted along the axis of abscissas and the signal intensity level is plotted along the axis of ordinates.

The results of the first series of experiments proved that when two complex sounds are simultaneously cut in they are received as some third resultant signal; the nature of the sound of this signal is determined by the relationship between their intensity levels. The problem which was tackled in the second series of experiments was the comparison of the possibilities of determining the nature (timbre) of the sound of the resultant signal under the conditions which obtained when the sounds that make up this signal were cut in simultaneously and when the component sounds were cut in with a time lag (one sound was cut in earlier, and then the second sound was added to it). In order to simplify the evaluation possibilities for the test subject we utilized complex signals which had characteristics that were close to the vowels ä and ē. These two sounds yield a third signal (when they produce a combined sound and when their levels are appropriately selected) which was defined by the majority of the test subjects as a sound similar to the vowel ĕ. The experiments were conducted in the following manner. First the dependency of the character of the responses (ä, ē or ĕ) upon the relationships between the levels of the component complex sounds was established for the simultaneous application of both sounds. Then the conditions were changed to those which obtain when the sounds are cut in with a time lag. The test subject was told in advance that one sound would appear first, that subsequently a second sound would be added to it and that from then on the two sounds would be received jointly. The test subject was to concentrate his attention upon the time during which the two sounds appeared together and determine which signal he heard (i.e., to which vowel the sound seemed most similar). Sounds were applied to the test subject in such a way that in half of the cases the sound began with the vowel ä and in the other half of the cases – with the vowel ē. The second sound was added 0.5-0.7 sec after the first sound had been cut in; then 1.0-1.3 sec after that both sounds were simultaneously cut out. The relationships between the levels of ä and ē were varied in random order in 5 db steps (this procedure was the same as that used in the experiments where the sounds were cut in simultaneously). Each relationship governing the levels was repeated 10 times during the experiment. The experiments were conducted on six test subjects of whom four had been given prolonged training in various kinds of auditory measurements. The aggregate results of the experiments on all six test subjects are shown in Fig. 6 [A) sounds cut in simultaneously, B) sounds cut in with a time lag]. From Fig. 6, A it is evident that when the sounds were cut in simultaneously, the dependence of the nature of the responses to the sound upon the relationship between the levels of the components of the sound is completely definite and regular. When the components corresponding to the vowel ē predominate the entire sound is recognized as the vowel ē; when the components which correspond to the vowel ä are predominant, the sound is recognized as the vowel ä. If the perception level for the components which represent the vowel ē exceeds the perception level of the ä components by 10-15 db, the sound is in the majority of cases recognized as the vowel ě.

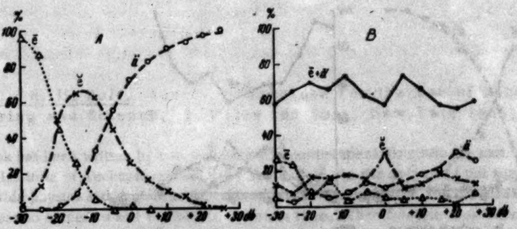


Fig. 6. Dependence of the character of the responses to the aggregate signal upon the relationship between the levels of the complex sounds which make up the signal (combined data obtained from six subjects). The difference in perception levels for complex sounds (the level of ä minus the level of ē) is plotted along the axis of abscissas; the responses to ä, ē and ě, expressed as a percentage of the over-all number of responses, is plotted along the axis of ordinates. A) both complex sounds are cut in simultaneously, B) one complex sound is cut in earlier than the other one.

A different picture was observed when the sounds were cut in with a time lag. In the majority of cases the test subjects responded that they perceived two different sounds: ä and ē, one of which sounded as if it were received against the background of the other vowel (Fig. 6, B, Curve ē + ä). In those cases when, in their opinion, the test subjects succeeded in hearing one resultant signal they could not determine the nature of its sound in a singular manner. The same sound was defined first as the vowel ē, then as the vowel ä and then as the vowel ě.

This can also be clearly seen on the individual curves for separate test subjects (Fig. 7). Some of them asserted that when the sounds were cut in with a time lag they were received most easily as one signal when they were close together in terms of their intensities (Fig. 7, B); others said that they could only hear one signal under conditions where one sound was considerably louder than the other (Fig. 7, B). In the latter case the impression may be created that the test subjects were evaluating the nature of the resulting sound fairly well, but the fact is that they were simply receiving the louder signal as a resultant sound and were ignoring the weaker signal.

The results of the experiments showed that depending upon how the complex sounds were cut in – simultaneously or with a certain time lag – they were either received as one signal or were separated into two independent signals. When the sounds were cut in simultaneously a person heard one signal whose timbre he could determine very precisely. The segregation of one of the component complex sounds from the aggregate perturbation proved to be extremely difficult in the case where the sounds were cut in simultaneously. Only one of the seven test subjects was successful in doing this in our experiments.

In the case where the sounds were cut in with a time lag the test subjects heard two independent signals, and the signal which appeared later in time could be recognized when its intensity was considerably lower than the intensity level of the first sound. Notwithstanding the fact that the two stimuli were received jointly over a very prolonged period (1.0-1.3 sec) after the second sound was added, the test subjects were not able to hear them as one composite signal – they were not able to determine the timbre of the resultant sound in a singular manner. These facts permit us to assume that the decisive role is played by temporal factors in the process of

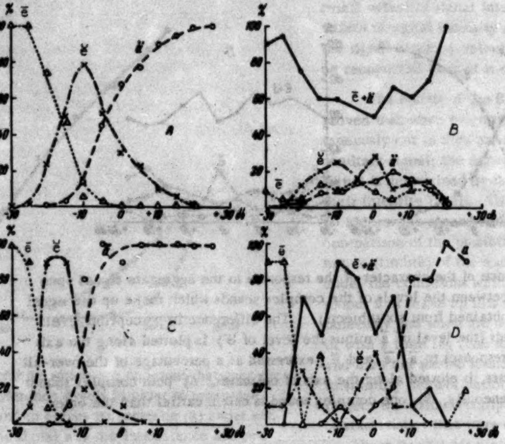


Fig. 7. Dependence of the nature of the responses to the aggregate signal upon the relationships between the levels of the complex sounds composing the signal; individual values were obtained for two persons. The notation is the same as in Fig. 6.

segregating a complex sound perturbation into its individual complex signals. A detailed investigation of the temporal characteristics of the process (i.e., the determination of that minimum time lag for which segregation into component signals can take place, the clarification of the significance of the rate at which the second signal is cut in, etc.) must evidently facilitate the clarification of those mechanisms by means of which this process is accomplished.

LITERATURE CITED

[1] R. Galambos, "Neural mechanisms of audition," Physiol. Rev. 34, 3, 497-529 (1954).

[2] G. Helmholtz, Study of Auditory Perceptions (1875).

I. P. Pavlov Physiology Institute of the
Academy of Sciences, USSR, Leningrad

Received May 18, 1955.

BIBLIOGRAPHY

T. F. Huter, R. H. Bolt. Sonics. Technique for the Use of Sound and Ultrasound in Engineering and Science, J. Wiley and Sons, New York 1955, 456 pages.

This book has been written by two well known acoustics specialists who are part of the staff of the Massachusetts Institute of Technology — T. Huter and R. Bolt. In the opinion of the authors the purpose of the book is to provide physics and engineering fundamentals for the scientific and practical applications of elastic waves, which are so numerous and varied at the present time. Therefore the book differs essentially from other fundamental texts on ultrasonics (for example, it differs from the encyclopedic summary of Bergman titled "Ultrasonics and its Application").

The book contains no systematic examination of problems which are directly related to one or another group of practical applications; practical applications are discussed only as random examples. The basic part of the book is devoted to a discussion of the general fundamentals of acoustics and to problems of designing individual units and elements in acoustic and ultrasonic equipment.

Chapter I is essentially an introduction. In the first part of this chapter the term "sonics" which has been introduced by the authors is defined. (They use this term as a title for the book.) The authors define this term as describing the sum total of the applications of sound in the solution of problems of measurement, monitoring and action upon matter. This chapter also gives a brief classification of applications which are known at the present time. The authors suggest that these applications be separated into three fundamental groups: a) applications for analytical purposes (which include the analysis of geometric dimensions, the measurement of elasticity constants and the investigation of molecular properties); b) the applications involving high-power ultrasonic signals, which in turn are subdivided into purely mechanical effects, chemical effects and biological effects; c) applications which use ultrasonic waves for monitoring (these applications include the monitoring of processes and systems, underwater signalling and communications) and special application (such as cavitation measurements, delay lines, the measurement of flow velocity, etc.). This classification, of course, does not exhaust this subject and only provides a general concept of the possible applications.

The second chapter contains a brief discussion of the fundamental concepts involved in the theory of vibration and propagation of elastic waves, excluding the problems of the radiation of sound which are segregated into a third chapter.

The fourth and fifth chapters, which are respectively devoted to piezoelectric and magnetostrictional transducers, are of considerable interest. It should be said that notwithstanding the very large number of papers which are available on these subjects these problems have not as yet been thoroughly clarified, and there is still no reasonably reliable method for solving the calculation and design problems which are involved here. In their book the authors have attempted to give a relatively orderly and complete system for the calculation and design of such vitally important devices as piezoelectric and magnetostrictional radiators and receivers, and they have collected the most essential known information. It cannot be said that they have completely solved this problem. But, in any case, these complex problems are discussed in a considerably more orderly fashion than anywhere else that we know of, and the material provided in the book permits the reader to fundamentally understand the physics involved in the operation of transducers and the methods by which they are designed.

In the sixth chapter the fundamentals of the physical properties associated with the effects of high-power elastic waves upon matter are given. After examining the difference between the structure of the field which is created by traveling waves, standing waves and a diffuse field, the authors examine the conditions required for obtaining high pressures and large accelerations. Then fundamental data concerning the effect of an acoustic field upon particles suspended in a gas or in a liquid are given. The concluding portion of the chapter is devoted to the problems of ultrasonic cavitation. Here the authors examine cavitational nuclei, noises created by cavitation, the vibration of cavitational bubbles, the cavitation threshold and the factors which determine it.

The seventh chapter provides a description of the basic sections which are used in ultrasonic devices (the emphasis is mainly on radiators). The chapter begins with the concepts involved in the selection of various types of radiators. Then methods are discussed for applying the ultrasonic waves to the substance being processed. Special attention is given to an examination of the methods for introducing ultrasonic waves into a living organism for therapeutic purposes. Following this the authors pass on to an examination of focusing systems — mirrors, lenses and concave radiators. Then the chapter examines the transmission of ultrasonic waves through metallic and ceramic rods which have both a constant and a variable cross section; the chapter also analyzes a method for introducing the ultrasonic waves into melts at high temperatures.

The second half of the chapter is devoted to electrodynamic and nonelectrical high frequency oscillators, to low frequency vibrators and low frequency cleaning machines. Here the description of direct hydrodynamic self-excited valve systems should be especially mentioned. The latter are of great interest, since they give promise of producing high-power ultrasonic vibrations by means of a simple and cheap method. In conclusion the chapter describes a low frequency system for boring holes, a special rotary generator which is utilized in paper production for homogenizing viscuous media, etc.

As a whole, the sixth and seventh chapters are of great interest both in terms of the general concepts contained there and in terms of the specific material which for the most part is being published for the first time.

The eighth chapter contains a summary of material relating to the application of sound in the capacity of a medium for monitoring and analysis. The authors propose to classify the various methods examined into the following four groups: a) methods which utilize standing waves (including resonant methods); b) methods which use traveling waves, c) methods which utilize measurement of reverberation processes or decay and d) pulse methods (both the reflection and "translucent" methods). Making use of this classification, the authors describe a great number of circuits which are utilized for measurement and monitoring.

In addition to the enumerated eight chapters the book contains an appendix which is devoted to relaxation processes in liquids and gases; this appendix provides a rather comprehensive and complete discussion of the approaches which exist at the present time, and supplies the most essential experimental data which applies to the problem under examination.

Thus the material available in the book brings the reader closer to the problems involved in the varied practical utilization of elastic vibrations. In this sense the value of this book is incontestable, since it contains that minimum of physical and procedural information which is necessary for an intelligent approach to problems which are being put forward by engineering practice.

One of the merits of this book is also the simplicity and clarity of its exposition and the good selection of examples illustrating practical applications. A number of these examples are taken from reports by various companies and laboratories, and these are being published widely for the first time.

The book is not free from small errors and inaccuracies (such as, for example, on p. 268, in calculating the amplification produced by concave radiators). But the number of such errors is small and their presence cannot alter the good impression which is produced by the book as a whole.

The book will undoubtedly be of great use to all those who are interested in or are working with the techniques of ultrasonics, which have become powerful tools for new research, for monitoring and for investigating the properties of matter.

L. D. Rozenberg

Allen E. Crawford. Ultrasonic Engineering. Butterworths Sci. Publ., London, 1955, 344 + X

The author of this book is the well-known English specialist Crawford who works in the laboratories of the "Mullard" Company. (This company is a pioneer in the industrial application of ultrasonics in Europe). Crawford has written his first monograph devoted to the application of high-power ultrasonic vibrations in industry. However, this monograph differs essentially from the recently published book by Huter and Bolt titled "Sonics".* If the latter book devoted its fundamental portion to the physical and physico-engineering aspects of the industrial application of ultrasonics and the actual specific applications were used only in the capacity of illustrative examples, then in the book by Crawford the contrary is true – the emphasis is placed upon the description of specific processes and devices.

The entire book is subdivided into three parts. The first part, consisting of two chapters, provides very brief reference data concerning the propagation of acoustic waves, their reflection and attenuation in various materials, as well as data on the measurement of sound intensity and frequency. As a whole, the first chapter gives the impression of insufficient thoroughness. Nothing is said in it concerning spherical waves (all of the data given refers to a plane wave), and nothing is said concerning such essential ultrasonic phenomena as diffraction, scattering etc. The second chapter devotes its first part to cavitation. In essence the basic material of this chapter is a reiteration of the well-known work by Willard. Granting the comprehensive and interesting article by Willard its full due, it is nevertheless impossible to acquiesce to the dominant position into which the author of this book places the article in relation to a great number of other investigations in this field. Willy-nilly one is here forced to compare this book with the book by Huter and Bolt where the problems of cavitation are analyzed from a considerably broader point of view.

The second portion of the book is devoted to the problems of ultrasonic generation and is subdivided into four chapters which respectively examine piezoelectric, magnetostrictional, hydrodynamic and electromagnetic radiators. The material is basically descriptive in nature and can in no way be a basis for the calculation or design of high-power industrial installations. However, it is true that the text provides a series of interesting photographs of specialized units and radiators, which are being published for the first time.

The third, most voluminous portion is devoted, as we have already said, to specific industrial applications; it is subdivided into seven chapters: coagulation and agglomeration, emulsification and dispersion, chemical applications, metallurgical applications, soldering and tinning of metals, the application of ultrasonics in biology and medicine and ultrasonic monitoring (including defectoscopy). This portion of the book is of great interest. It provides a great deal of material which generalizes the experiences both of the author himself and of other research workers. The material is systematized and is richly endowed with interesting illustrations and quantitative data. Thus, for example, in the chapter which deals with coagulation, schematics are given for various industrial gas-cleaning installations. The chapter on dispersion provides reference material which has been composed on the basis of data provided by various authors concerning the frequency selectivity of a process as a function of its degree of dispersion. The chapter on the chemical effects of ultrasonic waves provides examples of polymerization and depolymerization of the effects of ultrasonic waves upon crystallization, of the application of ultrasonics in electrochemistry, etc. . The chapter devoted to metallurgy examines the effect of ultrasonic waves upon melts and upon metals which are in a plastic state, the application of ultrasonic waves in arc furnaces, the effect of ultrasonic waves upon the processes of surface corrosion, diffusion and allotropic transformation in solid metals. In conclusion this chapter provides comprehensive data on the ultrasonic processing of solid metals. The chapter on the soldering of aluminum and light alloys is also comprehensively written.

In summary, it should be said that although Parts I and II are, in our opinion, not fully successful in their presentation, the fundamental material of the book is of interest as the first systematic reference description of the most important industrial applications of high-power ultrasonic devices. The book is, without a doubt, very useful for all those who are occupied with these problems.

L. D. Rozenberg.

---

* See p. 335 of this issue.

A LITTLE-KNOWN ARTICLE ON ACOUSTICS BY M. I. LOBACHEVSKII

ON THE HUNDREDTH ANNIVERSARY OF THE DEATH OF M. I. LOBACHEVSKII (1856-1956)

This year marks the hundredth anniversary of the death of the great scientist Nikolai Ivanovich Lobachevskii. Lobachevskii built the new geometric system – the non-Euclidian geometry which bears his name. This discovery produced a revolution in science, since it gave a new direction not only to geometry, but to mathematics as a whole and played a most important role in the problems involved in the theoretical foundations of modern physics.*

In December of 1955, while preparing the final volume of the complete collection of works by Lobachevskii for publication, I discovered a popular article titled "The origin and propagation of sound in the air" which appeared in the journal "Kazan Bulletin" in the year 1823.** This article has no signature and no end (at the end it said "continuation follows", but there never was any continuation). The concept which the author had concerning the mutual relationship between mathematics and natural science and other of his assertions, as well as his literary style in the article remove any doubts that this article was written by Lobachevskii. The thought which the article expresses can be uncovered, almost to the very word, both in his printed papers and in his manuscript materials which were written in the years 1822-1824.*** Lobachevskii's energetic style is so unique that his hand could be recognized ex ungue leonem from a mere glance at the article. This article was the first paper by Lobachevskii to be printed during his lifetime. Its full text is presented below.

THE ORIGIN AND PROPAGATION OF SOUND IN THE AIR

The causes which have prompted me to write this essay have been the new observations which French physicists have made of the propagation of sound, and a desire to make the Bulletin interesting to readers from the point of view that they will find truths in it which they might be curious to know – truths which have been hidden from them by the difficulties of computation. Those who are not professional mathematicians, or who know only the fundamentals of mathematics do not suspect the power and the wide scope of this subject, and perhaps doubt that by means of little experiments in a physics laboratory and a few calculations it is possible to see both how sound originates in the air and how great the velocity of this sound is; well, they will find proof of this here. Everything that has a magnitude belongs to the science of numbers; and what in the physical world does not have a magnitude ? Everything in the physical world exists under the necessary condition that it must be measurable, and therefore everything is subject to the laws of mathematics. Thus all of the natural sciences are striving to rise to that high level of perfection which will accrue to them when they are combined with mathematics; from the time of this combination their progress will go forward with rapid strides. This has already happened with physics and recently with minerology, and there is the hope that we can expect this for all of chemistry. Striving toward those truths which mathematics undover and whose importance amazes us, we shall find, if we return to the very source of great and unexpected successes, that rigorous science is at the same time the science of sound reasoning, and that the basis of everything is the valid understanding of things,which does not cease to lead mathematics through its calculations. Having said this, I imply that there is no phenomenon in nature which reasoning cannot clarify and that there are no phenomena which this reasoning cannot predict and for which it cannot determine both time and measure with accuracy. It would seem that an understanding of things and sound reasoning concerning these things should not depend upon calculations; it is nevertheless true that a mind trained in calculation continues far beyond those limits which a mind that is not initiated in the secrets of the science of numbers cannot transcend; thus, if I do not write for those who can plumb depths of this sort on the same level with me, then I am confident that I will be forgiven my expression in honor of mathematics,that I do not write for the many. It would be a vain attempt for mathematicians to decide to start from the very beginnings regardless of their scientific proficiency; otherwise why have calculations, if one can do without them? And so I shall limit myself solely to describing how physicists conceive the origin of sound, what serves as the basis for their calculations, whether practice justifies their theory and the rigorous nature of their calculations, and how far they can penetrate into the essence of things by the depth of their science.

---

* Concerning N. I. Lobachevskii and the significance of his discovery, see V. F. Kagan "Lobachevskii and His Geometry", Sketches for the Layman, State Tech. Press, Moscow, 1955. The geometric works of Lobachevskii together with comprehensive commentaries are contained in the recently published book: N. I. Lobachevskii, "Three Papers on Geometry", State Tech. Press, Moscow, 1956.

** Kazan' Bulletin No. VII, January 1823, pp. 49-60.

*** Comparisons will be given in the final volume of the complete works of Lobachevskii (now in preparation).

Sound is the motion of air and is very similar to the motion of waves; it is even more similar to the motion of the strings of musical instruments. As an example, the ignited powder flying out of the muzzle of a cannon disperses air into the surrounding area. The initial motion cannot be instantaneously imparted to the atmosphere; there must be a boundary defined by a rounded configuration beyond which the air remains unwavering at the instant of the explosion and to which the displaced air is gradually pressed. When the explosion is over the boundary of the initial motion, without requiring any additional force, again begins to expand and to thus transmit its motion further into the surrounding adjacent layer; this motion is completely analogous to the initial motion which, passing from layer to layer, carries sound in all directfons from the place from which it originates. The same thing occurs with a wave that is produced at the surface of water around a stone which has been cast: this wave expands and travels along the surface away from the spot at which the stone has fallen. A comparison of sound with waves can be still further supported. Two waves on the surface of the water cross when they meet; however after meeting the initial motion of each wave continues without any interference. In air two sounds come from different locations and each sound can be heard separately, independently of the other sound, without a loss in the intensity of either one. But here is the difference between the motion of air and the motion of waves. Many waves arise at the same time around a stone which has been thrown into the water: the largest wave is considerably more noticeable than the other waves, but the waves which precede and follow it can also be seen. They are separated into groups of circles which lie on the horizontal surface of the water and which travel together with the waves while lagging with respect to one another during the period of their motion (the waves which are enclosed between these circles can be viewed as one corrugated wave because their corrugated surface is not tangent to the level of the water). On the contrary, in air there is, as it were, one wave when there is only one impact similar to the impact of a stone falling into water: there is no motion on either side of the compressed layer.

With respect to strings the similarity goes further and can be called complete, with the exception of the fact that here the motion is propagated in only one direction and in air the motion is propagated in all directions. If a string vibrates in its entirety to either side when it emits a sound, it then represents the motion of air in wind instruments. If the length of the string is extremely great and the motion is produced in only one portion of it, it then represents the atmosphere and its vibration represents the motion of air during the period that the sound is present. Calculations show that when a string is drawn somewhere near its center and then released, the flexure produced in it is divided in two in a manner similar to a wave produced in water: one part moves in one direction and the other part moves in the opposite direction without leaving behind any motion and without being preceded by even any small vibration. Thus it is in air — the sound from a shot is propagated in all directions, produces a momentary impact when it reaches the ear, but is not audible before the impact and is not continued after the impact. Let us return again to the string. When the flexure reaches the spot where the end of the string is fastened, it is reflected back. In spacious atmosphere the sound is lost without reaching its limits, but it is reflected by projection from the earth and clouds in the air which, as it were, designate the places at which the air "string" is fastened. The difference between the motion of air and the motion which occurs in strings consists solely of the fact that the compressed air layers, while becoming continually more extensive, weaken the intensity of the sound, whereas the flexures of a string become weakened solely by extraneous causes (causes which dissipate any motion at the surface of the earth and finally force it to cease entirely). In cylindrical tubes, where the air layers cannot expand, the intensity of the sound is not attenuated. The French physicist Biot tested this in a water duct whose length was 951.25 meters.* The slightest whisper at one end could be clearly heard at the other end.

It is necessary to distinguish between the velocity of sound and the velocity of the intrinsic motion of the air. Let us resort to an analogy and again take the example of the movement of waves on the surface of water. Observing the fluctuating surface of the water, it is easy to notice that whereas one wave travels rapidly after another, the floating corrugated structure moves forward very slowly. The waves are only the ripple of the water, an alternate rising and falling, while the corrugated structure as a whole is carried by the intrinsic motion of the water in the river. If it becomes noticeable to the eye that the corrugated structure moves forward as a whole, then this is caused either by the wind or by the flow of water which is due to the inclination of the river bed, but such motion could hardly be noticed if it were produced solely by wave excitation. This fact has been used in mathematics in order to simplify the calculations both for the motion of waves and for the motion of air. The gradualness of intrinsic motion and the fact that this motion is in various directions produce a deviation from the main formation of waves; this deviation consists of a slight lateral vibration of the surface of the main wave. Applying this to air, physicists use this basis to argue the point concerning a difference in sound (timbre) in various sounds and a difference in pronunciation produced by a human voice. Mathematicians do not follow these shadings of sound since their science is not equipped to go this far: they calculate only that which relates to the propagation of sound and to the difference between musical tones.

A "sound-producing" wave (I wish to use this term to define the vibrating portion of the air during the time that the sound is present) retains an identical wavelength during the entire time that it is in motion. Meanwhile the velocity of the intrinsic motion of the air particles (the intensity of the sound depends upon this) must become the smaller, the further the wave travels. The air particles which comprise "the sound-producing" wave oscillate back and forth in a manner similar to a clock pendulum which swings from one side to the other. The greatest possible wavelength of a "sound-producing" wave for which an effect is produced upon the perception of the human auditory system is a wavelength of 10.42 meters; the smallest wavelength which can produce an auditory effect is a wavelength of 40.7 millimeters. Beyond these limits the motion of the air is either indistinguishable from wind noises or is imperceptible to the ear because of its minute magnitude.

Sound is a general definition for any motion of the air which is perceptible to the ear. Such a motion is produced by a shot – the vibration of the air is instantaneous and occurs only once. A sound is also a tone which is produced when several displacements follow one after another and generate waves that have an identical wavelength (such an example, is a "sounding" string); however if the wavelengths of the successive waves are different, the sound will be received as noise. The "sounding" waves of a tone vibrate from one side to another in the same manner as do the strings which produce them. A definite wavelength corresponds to a definite time which is required for one complete vibration of the string. Thus the tone is audible when the impacts of the air upon the ear are repeated at equal time intervals. The time which elapses between tonal impacts is extremely short. The lowest tone which can be heard by the ear beats 30 times per second. The highest tone which can be heard by the ear beats 12,000 times per second. In the low tones one can still hear something which is similar to repeated impacts, but in general a tone which can be heard produces the same effect upon the ear as a red-hot coal in a brisk blast of air produces upon the eye: a continuous bright circle is seen.

---

* Different peoples have different units of measurement. In order to circumvent mathematical deviations we have agreed to utilize only one measure in our books – meters and fractions of a meter. Such a measure is not subject either to variations or to loss. We do not know exactly how large a measure was represented by the Greek Stage. The measures of presently existing peoples vary continually in the same way that monetary values vary. The Egyptians cut out images on obelisks to retain the constancy of measures, but sometimes obelisks are destroyed. Mathematicians have conceived a measure which is indelibly printed in nature and which will only be lost when education dies, but it will again be found when mathematics is reborn.

The intrinsic motion of the air particles depends upon the initial displacement and is therefore arbitrary; however it is still enclosed within certain limits. It is possible to assume that when initial deviations of air particles occur which are 1/10 of a vershok* in magnitude, such a sound will still be audible at a distance of four versts**; in contrast to this the air will hardly move over a distance of 1/100,000 of a vershok. Our hearing is extremely acute. Is it any wonder that at night we hear the rustle of an insect upon the wall or the falling of grains of sand ?

I. N. Bronshtein

(Continuation to follow)

---

* (1 vershok = 40.7 mm).

** 1 verst = 1.0668 km.

LETTERS TO THE EDITOR

A CYLINDRICAL RADIATOR MANUFACTURED FROM BARIUM TITANATE CERAMIC, WHICH RADIATES ALONG THE AXIS OF THE CYLINDER

A. A. Anan'eva

As we know, it is possible to incline the directivity characteristic of a radiator with respect to the radiating surface by controlling the phase variation along the vibrating surface of the radiator in a definite manner. In particular, for "linear" radiators (for example, for a cylindrical tube) it is possible to obtain a directivity characteristic whose maximum is directed along the line of radiation [1-4].

This present paper describes a cylindrical radiator which consists of a one-piece tube, made out of barium titanate ceramic, upon whose surface a distribution of normal velocities is created along the generant of the cylinder; this distribution is such that it corresponds to the standing-wave law [5]. The field created by such a radiator in the surrounding medium is the sum of two spectra of cylindrical waves that are inclined at an angle $\pm\alpha$ with respect to the plane which is perpendicular to the axis of the cylinder; this angle is determined by the relationship.

$$\sin\alpha = \frac{\lambda}{d},$$

where λ is the sound wavelength in the medium and $\underline{d}$ is the periodicity of the distribution of the normal velocities. The alternating (in sign) distribution of the normal velocities was created by means of the opposite polarization of intermediate sectors with a length $d/2$; this was achieved by using sectioned external electrodes and a solid internal electrode. The same electrodes, arranged in a different commutative manner, are also utilized for feeding a ready-made radiator. Figure 1, a shows the external view of the ceramic tube, and Fig. 1, b shows the radiator mounted together with the tube (the tube is covered with water-repellent lacquer). The specified radiator is designed in such a way that the maximum radiation (this radiation corresponds to the frequency of fundamental radial resonance – 40 kc) is directed along the axis of the cylinder (the case $d = \lambda$). The length of the tube used for the radiator was equal to 10λ. When the radiator is in operation the metallic rings on the external surface of the cylinder (which are connected by a common bus-bar) serve as its grounded electrode. Hydro-insulation was achieved for the internal electrode and the cable lead-in by means of a sealing unit – a gland located at one end of the tube and comprising part of the mechanical support which fixes the radiator in a specified position in space. The diameter of the flange for the gland was approximately equal to two wavelengths of the soundwave in water (at an operating frequency of 40 kc); in blocking the radiation which travels in one direction, the flange reflected it in the direction of the free end of the tube. As a result the effective length of the radiator was doubled, and this permitted us to obtain relatively narrow directivity characteristics at low frequencies. Figure 2 shows examples of directivity characteristics in

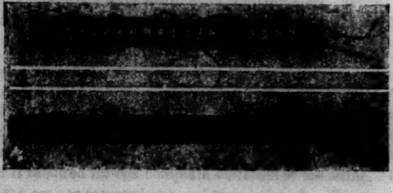
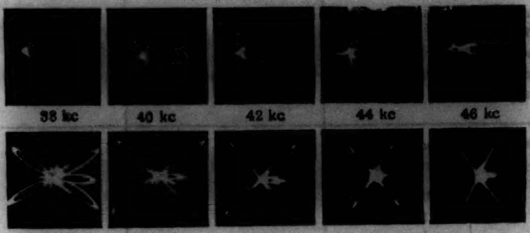


Fig. 1. a) Ceramic tube with ring electrodes, b) radiator in assembled form.

the axial plane of the tube for various radiation frequencies; in the plane perpendicular to its axis the transducer was nondirectional because of the cylindrical symmetry of its construction. Figure 3 shows a comparison between the calculated values of the angle α which correspond to various frequencies and the experimental values which are obtained from the directivity characteristics. The measurements of the intrinsic frequencies of the transducer showed that it had two operating frequencies: the first was a low-frequency resonance of radial vibrations and was determined in air by the relationship $\pi D = \lambda_1$, (D is the average diameter of the tube, λ_1 is the wavelength of the sound in the ceramic), and the second frequency is the high-frequency resonance of longitudinal vibrations which is determined by the condition $\lambda_1 = d$. When the device is operated in water the frequency which is determined by low-frequency resonance is lowered by approximately 19%, whereas the high-frequency resonance is only displaced by 2% in terms of its frequency. When the device is operated at the low-frequency resonance point the transducer exhibits an increased radiation impedance (the impedance is increased by a factor of approximately 1.6) in comparison to the case when the in-phase surface vibrations are created.


38 kc 40 kc 42 kc 44 kc 46 kc

55 kc 60 kc 70 kc 80 kc 90 kc

Fig. 2. Directivity characteristics for the radiator in its axial plane.

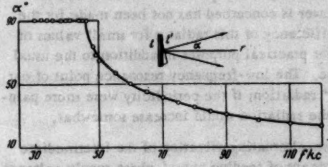


Fig. 3. Angle of inclination for the maximum of the directivity characteristic with respect to the plane which is perpendicular to the axis of the cylinder. Solid line – calculated values; light circles – points representing experimental results.

We made preliminary measurements of the electromechanical parameters of the specified radiator for various of its operating regimes: unloaded (in a vacuum), in air and in water; the acoustical-mechanical efficiency was calculated according to the well-known method which utilizes resonance curves. In order to determine the electromechanical efficiency we conducted measurements of the electrical losses while the transducer was operating outside of resonance. A knowledge of the two efficiencies permitted us to determine the electroacoustical efficiency. By conducting additional measurements of the resistive component of the electrical impedance of the ceramic tube while it operated in a vacuum without any mechanical supports, it was possible to segregate the mechanical losses in the supports from the remaining mechanical losses; it was thus possible to evaluate the part played by the internal mechanical losses in the ceramic.

The table provides preliminary results from the measurements of the transducer parameters; these measurements were made at small values of electrical power.

TABLE

Type of vibrations / Measured quantities		Radial resonance $f_1 = 40$ kc	Longitudinal resonance $f_2 = 136$ kc
Electromechanical efficiency		82%	98%
Mechanical efficiency		73%	48%
Electroacoustical efficiency		60%	47%
Power coefficient in water		0.13-0.14	0.787
Electrical losses		17.5%	2%
Mechanical losses	Losses in the supports	2%	10%
	Internal mechanical losses	20.5%	41%

The lateral spectra which are directed at an angle of ± 19° with respect to the plane which is perpendicular to the axis of the cylinder must correspond to radiation at the longitudinal resonance frequency. When the radiator operates at that frequency the radiation impedance is less than for radiation along the axis of the cylinder. This explains the relatively large internal losses at this frequency as compared to the losses which occur when the radiator operates at the radial resonance frequency. It should be noted that the electrical impedance of the radiator at the longitudinal resonance frequency has a relatively small reactive component in comparison to its resistive component. In view of the fact that up to this time the given transducer has only been tested at small values of electrical power (up to 0.6-0.7 watts/cm^2), a comparison between it and the commonly used ceramic transucers as far as maximum radiated power is concerned has not been made for the time being. The results of the preliminary measurements of the efficiency of this radiator for small values of power permit us to recommend this type of sound wave radiation for practical purposes in addition to the usual ceramic radiators which operate with surfaces that vibrate in phase. The low-frequency resonance point of our model did not exactly coincide with the frequency of the "sliding" radiation; if the periodicity were more painstakingly selected, it might be expected that the effectiveness of the radiation would increase somewhat.

In conclusion let us note that we could have achieved a phase-alternating vibration of the intermediate sectors (instead of alternating polarization of the tube sectors) by means of feeding out-of-phase supply voltages to the corresponding electrodes; in doing so we could have retained the same system of electrodes and utilized radial polarization of the same sign. Such a system provides certain advantages in those cases when it is necessary to utilize a dc field for polarization "bias".

I take this opportunity to express my appreciation to G. B. Maliuzhinets and to M. A. Isakovich for their valuable advice; I also wish to take note of the participation of V. A. Basova and A. M. Saprygina in the work involved in making the measurements.

LITERATURE CITED

[1] D. T. Laird and H. Cohen, "Directionality patterns for acoustic radiation from a source in a rigid cylinder," J. Acoust. Soc. Am. 24, 46-49 (1952).

[2] M. C. Junger, "The physical interpretation of the expression for an outgoing wave in cylindrical coordinates," J. Acoust. Soc. Am. 25, 40-47 (1952).

[3] D. H. Robey, "On the radiation impedance of an array of finite cylinders," J. Acoust. Soc. Am. 27, 4, 706-710 (1955).

[4] G. J. Thiessen, "On the efficiency of an acoustic line source with progressive phase shift," Can. J. Phys. 33, 11, 618-621 (1955).

[5] A. A. Anan'eva, Testing a Ceramic Radiator Which Operates According to the Traveling-Wave Principle (Report of the Acoustics Institute of the Acad. Sci. USSR, 1955).

Acoustics Institute of the
Acad. Sci. USSR, Moscow.

Received March 26, 1956

SECOND APPROXIMATION ACOUSTIC EQUATIONS AND THE PROPAGATION OF PLANE WAVES OF FINITE AMPLITUDE

Z. A. Gol'dberg

In analyzing acoustical problems which involve second approximations, the usual procedure is to base the analysis upon the equations of hydrodynamics (see, for example, [1]):

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v}\nabla)\mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} + \left(\frac{\eta}{3} + \zeta\right) \nabla \operatorname{div} \mathbf{v}, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0, \tag{2}$$

and the energy equation,

$$\rho T \left(\frac{\partial s}{\partial t} + \mathbf{v}\nabla s\right) = \sigma'_{ik} \frac{\partial v_i}{\partial x_k} + \operatorname{div}(\varkappa \nabla T), \tag{3}$$

which are solved by the method of successive approximations. A method for deriving second approximation acoustical equations was given in [2] where the equations of hydrodynamics were examined while taking viscosity into account. The process of sound propagation was considered to be adiabatic.

In this present paper a refinement is made of the solution of the problem involving the rigorous derivation of second approximation acoustical equations (this problem was posed by Eckart [2]), and in this regard an analysis is made of when the process of sound propagation can be considered to be adiabatic in problems involving a second approximation and the propagation of plane waves of finite amplitude.

The equation which defines the relationship between small variations in pressure p', density ρ' and entropy s' can be assumed to be of the form

$$p' = c^2 \rho' + \left(\frac{\partial p}{\partial s}\right)_\rho s' + \frac{1}{2}\left(\frac{\partial c^2}{\partial \rho}\right)_s \rho'^2 + \dots, \tag{4}$$

since it is natural to assume that the quadratic terms of the expansion containing s' are much smaller than the term $\left(\frac{\partial p}{\partial s}\right)_\rho s'$.* Here $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$. It is usually assumed that the process of wave propagation is adiabatic (i.e., in Equation (4) the term $\left(\frac{\partial p}{\partial s}\right)_\rho s'$ is neglected in comparison to $c^2 \rho'$, $\frac{1}{2}\left(\frac{\partial c^2}{\partial \rho}\right)_s \rho'^2$; however we shall evaluate the ratio between the terms $\left(\frac{\partial p}{\partial s}\right)_\rho s'$ and $c^2 \rho'$.

---

* We shall agree to call terms which contain s' thermal terms.

Let us examine the case when the absorption of sound is small over a wavelength:

$$\frac{\gamma}{k} \ll 1, \tag{5}$$

where γ is the absorption coefficient, $k = \frac{\omega}{c}$ is the wave number; i.e.,

$$\frac{\omega\eta\left(\frac{4}{3} + \frac{\zeta}{\eta}\right)}{\rho c^2} \ll 1 \tag{6}$$

and

$$\frac{\omega\varkappa\left(1 - \frac{c_v}{c_p}\right)}{\rho c_v c^2} \ll 1. \tag{7}$$

Here κ is the coefficient of thermal conductivity, c_V and c_P are the specific heats at constant volume and constant pressure. For the majority of substances $\eta \sim \frac{\kappa}{c_V}$ obtains (for more detail see [1]; § 77) $\left(1 - \frac{c_V}{c_P}\right) \sim 1$ and thus Equations (6) and (7) are of the same order of smallness.

Writing the equation which relates p', ρ' and the temperature variation T', and then comparing it with Equation (4), we obtain

$$\left(\frac{\partial p}{\partial s}\right)_\rho s' = \left[\left(\frac{\partial p}{\partial \rho}\right)_T - c^2\right]\rho' + \left(\frac{\partial p}{\partial T}\right)_\rho T'. \tag{8}$$

Finding the dependence of T' upon ρ' in a plane wave from the linearized energy equation and substituting it into Equation (8), we obtain

$$\frac{(\partial p/\partial s)_\rho\, s'}{c^2\rho'} = \left(1 - \frac{c_v}{c_p}\right)\frac{\varkappa\omega}{\rho c_v c^2}. \tag{9}$$

after dividing the left and right sides by $c^2\rho'$. In performing the calculations we neglected the terms which were less than the terms used above by a factor of γ/k. Relationship (9) will be taken into account when the first and second approximation equations are written.

We shall select certain units: length – X cm, time – τ sec., velocity – U cm/sec., density (the density of the unperturbed medium) – $\rho_0\ \frac{\text{g}}{\text{cm}^3}$. In dimensionless coordinates * the original equations can be reduced to the form:

$$N\rho\frac{\partial \mathbf{v}}{\partial t} + N^2\rho(\mathbf{v}\nabla)\mathbf{v} = -\overline{c^2}\nabla\rho' - \frac{1}{2}\left(\frac{\overline{\partial c^2}}{\partial \rho}\right)_s \nabla\rho'^2 + \frac{N^2}{R}\nabla^2\mathbf{v} + \frac{N^2}{R}\left(\frac{1}{3} + \frac{\zeta}{\eta}\right)\nabla \operatorname{div}\mathbf{v}, \tag{10}$$

$$\frac{\partial \rho}{\partial t} + N \operatorname{div}\rho\mathbf{v} = 0. \tag{11}$$

---

* The dimensionless coordinates shall be designated by a horizontal line which appears above the appropriate letter (however, this will be done only where they can be confused with coordinates which have a dimension).

Here
$$N \equiv \frac{U\tau}{X}, \ \nu \equiv \frac{\eta}{\rho_0}, \ \overline{c^2} \equiv \frac{c^2\tau^2}{X^2}, \ R \equiv \frac{XU}{\nu}.$$

In order that the selected system of units should serve the purpose of deriving the acoustical equations of the first and second approximation, it is necessary that the dimensionless derivatives of any variables (for example, ρ) be of the same order of magnitude:

$$\frac{\partial\bar{\rho}}{\partial\bar{t}} \sim \frac{\partial\bar{\rho}}{\partial\bar{x}} \sim \frac{\partial^2\bar{\rho}}{\partial\bar{x}^2}, \tag{12}$$

where $\underline{x}$ is one of the spatial coordinates. For a sinusoidal wave process Conditions (12) will be fulfilled if

$$X/\tau \sim c, \quad kX \sim 1. \tag{13}$$

We shall assume that the Relationships (13) are also fulfilled for acoustic processes which are close to being sinusoidal. We shall choose the unit velocity in such a way that

$$|\mathbf{v}|/U \sim 1, \text{for example } U = |\mathbf{v}|_{\max}. \tag{14}$$

Let us examine the case U << c [this means that

$$N \sim U/c \ll 1] \tag{15}$$

We seek dimensionless hydrodynamic coordinates in the form of a power series of N:

$$\begin{aligned} \bar{\rho} &= 1 + N\bar{\rho}_1 + N^2\bar{\rho}_2 + \ldots, \\ \bar{\mathbf{v}} &= 0 + \bar{\mathbf{v}}_1 + N\bar{\mathbf{v}}_2 + \ldots \end{aligned} \tag{16}$$

Before deriving the first and second approximation equations it is necessary to answer the question: of what order of magnitude is the expression N^2/R * with respect to N [see Equation (10)] ? In the paper by Eckart such a question does not arise since he only requires one condition instead of the two Conditions (12). As a result the X unit is indeterminate. He selects it in such a way that $N^2/R = N$.

Taking Relationship (13) into account we obtain the result that

$$\frac{N^2}{R} \sim N\cdot\frac{\nu\omega}{c^2}. \tag{17}$$

Therefore we must distinguish between three cases:

$$N \ll \frac{\nu\omega}{c^2}, \text{ i.e. } \frac{N^2}{R} \gg N^2; \tag{18}$$

$$N \sim \frac{\nu\omega}{c^2}, \text{ i.e. } \frac{N^2}{R} \sim N^2; \tag{19}$$

---

* It would be more accurate to speak of the expression $\dfrac{N^2}{R}\left(\dfrac{1}{3} + \dfrac{\zeta}{\eta}\right)$.

$$N \gg \frac{\nu\omega}{c^2}, \text{ i.e. } \frac{N^2}{R} \ll N^3. \tag{20}$$

Each case will be described by its own particular first and second approximation equations. The ratio between the term containing the viscosity and the term $c^2 \nabla \rho'$ in Equation (10) proves to be equal to a magnitude of the order of $\frac{\nu\,\omega}{c^2}$. Comparing this result with the results obtained from Formulas (6), (7) and (9), we conclude that for the majority of substances the thermal terms and the terms containing the voscosity are of the same order of magnitude *

$$N, \frac{\left(\frac{4}{3}\eta + \zeta\right)\omega}{\rho c^2} \text{ and } \frac{\varkappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\omega}{\rho c^3}.$$

Further mathematical treatment in the first case leads to the equations obtained by Eckart, but with the additional term $\left(\frac{\partial p}{\partial s}\right)_\rho \nabla s_1$ appearing in the first approximation equation of motion and with additional second order thermal terms appearing in the second approximation equation of motion. In the second case the viscous and thermal terms having the index one (and also being of the first power) appear in the second approximation equation. In the third case it is not necessary to write the viscous and thermal terms in the second approximation equations.

Let us examine a plane traveling wave. Eliminating ρ_2 from the second approximation equation of motion, we obtain

$$\frac{\partial^2 v_2}{\partial t^2} - c^2 \frac{\partial^2 v_2}{\partial z^2} - \frac{\left(\frac{4}{3}\eta + \zeta\right)}{\rho_0} \frac{\partial^3 v_2}{\partial t\, \partial z^2} = f(\rho_1, v_1), \tag{21}$$

in the first case; in the third case we obtain

$$\frac{\partial^2 v_2}{\partial t^2} - c^2 \frac{\partial^2 v_2}{\partial z^2} = f_1(\rho_1, v_1), \tag{22}$$

where $f(\rho_1, v_1)$ and $f_1(\rho_1, v_1)$ are certain functions of ρ_1 and v_1. The particular solutions of Equations (21) and (22) are of the form

$$v_2 = [A_1 \sin 2(kz - \omega t) + A_2 \cos 2(kz - \omega t) + A_3]\, e^{-2\gamma z}, \tag{23}$$

$$v_2 = A(z + ct) \cos 2(kz - \omega t). \tag{24}$$

The latter equation coincides with the second approximation of the Riemann solution. This evidently means that, notwithstanding the presence of the viscosity, the shape of the wave has a tendency to accrue a discontinuity. An analogous result is also obtained in the second case. Therefore the value of the parameter [see Equations (18-20)]

$$\frac{N}{\frac{\nu\omega}{c^2}} \approx \frac{p_1}{\eta\omega} \sim 1$$

---

* In the general case the magnitude of the thermal terms as compared to the remaining terms is determined by the relationship between the parameters.

can serve as a rough criterion * for the possibility of accrueing a discontinuity:

$$\frac{p_1}{\omega\left[\frac{4}{3}\eta+\zeta+\varkappa\left(\frac{1}{c_v}-\frac{1}{c_p}\right)\right]}.$$

Let us calculate the increase in the absorption coefficient of a plane wave, where this increase is caused by the existence of a second harmonic under the conditions which obtain in the first case. Making use of the method indicated in the book cited in [1], Chapter 77 and the result obtained from Equation (23), we obtain

$$\frac{\gamma_1}{\gamma}=1+\left[1+\frac{\rho_0\left(\frac{\partial c^2}{\partial\rho}\right)_s}{2c^2}\right]^2\cdot\left[\frac{p_1}{\left(\frac{4}{3}\eta+\zeta\right)\omega}\right]^2. \qquad (25)$$

In the process of the calculations we neglected terms which were less than the terms written by a factor of γ/k (as well as thermal terms) in comparison to the viscuous terms.

Formula (25) was obtained for the case when $\frac{p_1}{(4/3\eta+\zeta)\omega}\ll 1$; however we can also expect this formula to satisfactorily describe the absorption when $\frac{p_1}{(4/3\eta+\zeta)\omega}<1$.

In the second and third cases the shock waves which are formed lead to a considerable additional energy dissipation [1].

I express my deep appreciation to N. N. Andreev and I. G. Shaposhonikov for their interest in and discussion of this paper.

LITERATURE CITED

[1] L. D. Landau and E. M. Lifshits, Mechanics of Solid Media, (State Tech. and Theor. Lit. Press, Moscow, 1954).

[2] C. Eckart, "Vortices and streams caused by sound waves," Phys. Rev. 73, 1 (1948).

Magnitogorsk State Pedagogical Institute
Received March 16, 1956.

*** In the general case this parameter is of the form:**

$$\frac{p_1}{\omega\left[\frac{4}{3}\eta+\zeta+\varkappa\left(\frac{1}{c_v}-\frac{1}{c_p}\right)\right]}.$$

SOVIET PHYSICS – ACOUSTICS VOL. 2, NO. 3 July – August – September, 1956

(Published by the American Institute of Physics May, 1957)

CONTENTS

	Page	Russian page
Propagation of Sound in Inhomogeneous Media. L. M. Brekhovskikh	247	235
Instrument for Measuring the Elastic Moduli of Crystals. K. S. Aleksandrov and O. V. Nosikov	256	244
A Modulation Method for Measuring Small Voltages in the Audio-Frequency Range. D. K. Balabukha, L. L. Miasnikov and E. N. Plotnikova	260	248
The Problem of New Coefficients for Evaluating the Quality of Room Acoustics. E. E. Golikov	267	255
Calculation of the Radiation Resistance of Several Types of Distributed Radiator Systems. M. I. Karnovskii	280	267
Correlation Method for Measuring the Distortion Coefficient of Transmission. M. A. Sapozhkov	294	279
The Theory of an Ultrasonic Interferometer. V. A. Solov'ev	301	285
Propagation of Elastic Waves in a Medium with Cylindrical Channels. V. V. Tiutekin	307	291
The Effect of the Presonic Flow Velocity Upon the Radiation Impedance of a Piston In An Infinite Baffle. D. N. Chetaev	319	302
The Significance of Temporal Factors in the Reception of Complex Sounds. L. A. Chistovich	328	310

BIBLIOGRAPHY

	Page	Russian page
T. F. Huter, R. H. Bolt. Sonics. Technique for the Use of Sound and Ultrasound in Engineering and Science. L. D. Rozenberg	335	317
Allen E. Crawford. Ultrasonic Engineering. L. D. Rozenberg	337	318
A Little-Known Article on Acoustics by Lobachevskii. M. Lobachevskii – I. N. Bropshtein	338	320

LETTERS TO THE EDITOR

	Page	Russian page
A Cylindrical Radiator, Manufactured From Barium Titanate Ceramic, Which Radiates Along the Axis of the Cylinder. A. A. Anan'eva	342	323
Second-Approximation Acoustic Equations and the Propagation of Plane Waves of Finite Amplitude. Z. A. Gol'dberg	346	325

Leningrad State Pedagogical Institute

---

* In the general case the parameter [illegible] of the form

The expanded program of the American Institute of Physics comprises translation of four leading Soviet physics journals, as listed below. These translations, by competent, qualified scientists, will provide all research laboratories and libraries with accurate and up-to-date information of the results of research in the U.S.S.R.

Soviet Physics — Technical Physics

A translation of the "Journal of Technical Physics" of the Academy of Sciences of the U.S.S.R. The 1956 volume, which was originally announced as containing eighteen issues, is instead being issued monthly, in expanded issues. All issues of the original Russian journal for 1956 will be translated. Approximately 4,000 Russian pages. Annually, $90.00 domestic, $95.00 foreign.

Soviet Physics — Acoustics

A translation of the "Journal of Acoustics" of the Academy of Sciences of the U.S.S.R. Four issues per year, approximately 500 Russian pages. Annually, $20.00 domestic, $22.00 foreign. The 1955 issues of "Journal of Acoustics" U.S.S.R. will also be published. Will consist of two volumes, approximately 500 pages, and the subscription price will be $20.00 for the set.

Soviet Physics "Doklady"

A translation of the "Physics Section" of the Proceedings of the Academy of Sciences of the U.S.S.R. Six issues per year, approximately 900 Russian pages. Annually $25.00 domestic, $27.50 foreign.

Soviet Physics — JETP

A translation of the "Journal of Experimental and Theoretical Physics" of the Academy of Sciences of the U.S.S.R. Twelve issues per year, approximately 2,600 Russian pages. Annually $60.00 domestic, $64.00 foreign. Back issues of Volume 1 and 2 are available at a special price of $30.00 for both to subscribers to Volume 3.

Subscriptions should be addressed to the American Institute of Physics, 57 East 55th Street, New York 22, N.Y.